Lecture 1: MAT670

Comment
: more breaks off
Definitions...

1.0) Admin

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Room ??? hopefully NT020008

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Evaluation: • Short quizzes
• a project: 20^+ minute talk
+ written summary

I can update via email, or the TU Online...
— wkusner.github.io/MAT670/

We will take a 10 min break at ~ 15:50

If we end early / I run out of material...
we can stop.

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1.1 Mathematics...

Packings, Lattices and Configurations

Classical
Convex/
Discrete
Geometry

Applied Topology
Morse Theory

Geometry
of
numbers...

Rigidity

Stat Mech
Combinatorics
information theory

Various ideas from each, all ~~to~~ dealing with configurations...

{ Rigidity Theory, Configurations of linkages
{ Information Theory

Geometry of Numbers

Packing Problems ← my main motivation.

⋮

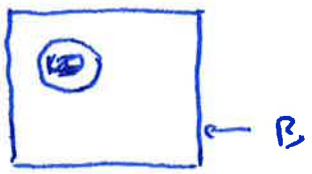
Easy to state, hard to solve.

≡ [Packing, inequalities
[configuration spaces...

1.2 Packing problems.

Start with 2D

packing circles (Disks) in a box.



Want to maximize density

(volume fraction) $\dfrac{\text{Volume}(K)}{\text{Volume}(B)}$

Now, for $\bigcup_{i=1}^{N} K_i$

$K \sim K_i \sim K_j \quad (K_i = \varphi K_j) : \varphi \in \mathbb{R}^2 \rtimes SO2$

$K_i \subset B$

$\mathring{K}_i \cap \mathring{K}_j = \varphi$

Congruent, contained in Box, it is a packing.

e.g. consider the largest possible ~~radius~~ scale r

st $\bigcup_{i=1}^{N} \varphi_i r K_i \subset B$, is a packing of B

{Minkowski notation}

or, find a configuration $\{X\}$ ~~collection~~ of

N points in B st

$d(x_i, x_j) > r$ for $i \neq j$

and $d(x_i, \partial B) \geq r \ \forall i$

These are hard problems.

infinite variety by changing

N, K and B, dimension...

adding other constraints...

One case of interest is the $B = \mathbb{R}^2$ case...

The circle packing problem in the plane.

or... $B = \mathbb{R}^d$

or $B = S^2$

1.2) $B = \mathbb{R}^2$

We have an infinite collection of congruent Disks, now normalized to have unit radius ~~(~~ $\mathring{K}_i \cap \mathring{K}_j = \varphi$

The density of such a packing P (upper) is given by:

$$\delta^+(P) = \limsup_{r \to \infty} \frac{\mathrm{Vol}(P \cap r\mathbb{D}^2)}{\mathrm{Vol}(r\mathbb{D}^2)}$$

Similarly, Lower density...

These quantities may not be nice ~~well defined~~ in general...

- does not depend on $\vec{0}$
- does depend on $\mathbb{D}^2$ shape.

For general... $\delta^+(P) = \ldots \frac{P \cap D^n}{D^n}$

Does such a packing exist? ...

=

We might look at this in more detail later ...

=

How can we compute bounds on ~~the~~ density?

Clearly, $\delta^{+}(P) \le 1$

= (bad...)

Also, we can construct lower bounds using highly structured packings –

For example.

$(2\mathbb{Z})^d$ is a packing.

Since this structure is periodic... we can work with the fundamental domain.

Easy Exercise: (why..?)

$$V_d(~~\text{vol}~~\ S^d(r)) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} r^d$$

Ball

easy in even dim.

$$V_{2d} = \frac{\pi^{d}}{d!}\, r^{2d}$$

density

$$\frac{\frac{\pi^d}{d!}}{2^{2d}} = \left(\frac{\pi^{\frac{d}{2}}}{(\frac{d}{2})!\, 2^d}\right)$$

We could improve this by considering better ~~lattic~~ periodic systems

[Easy exercise: A ~~best~~ periodic system can approximate a densest packing.]

but ...

[Consider ~~the~~ A' $A_1 \times A_1 \ldots A_d$?
root systems..]

1.3

Better Bound (non-constructive)

Consider our packing to be saturated.

That is, P st $P \cup_{\varphi} K$ is not a packing for any φ in $\mathbb{R}^n \times SO(n)$

- No additional disks can be added.
- 1. they certainly exist (infinite algorithm..)
- 2. ~~They have~~ saturation does not decrease $\delta^+(P)$

Surprisingly useful idea...

P Saturation $\Rightarrow d(x, K_i) < 1 \quad \forall x \in \mathbb{R}^n$

$\Rightarrow$ ~~2K~~ $\{2K_i\}$ is a covering.

~~to~~ double.

$\boxed{(?\, \varphi_i^{-1} 2K)\, \varphi_i}$

Correct...

$\Rightarrow 2^d \cdot \delta^+(P) \geq 1$

$\Rightarrow \delta^+(P) \geq \dfrac{1}{2^d}$

We may go over some slight improvements in the future... but none seem satisfactory

Problem is that the ~~local structure~~ there is a lot more freedom in $d \geq 3$! even.

$\Big[(2\mathbb{Z})^4 \ldots \quad \Delta = 2\sqrt{4} = 4 \ldots$

1.4) upper bounds in $d = 2$.

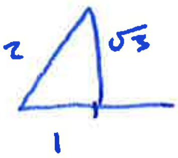
~~There is a~~

1) The lower bound is constructive.

hex, Δ lattice

$$\delta = \frac{\pi}{\sqrt{12}}$$

equilateral Δ. base 2



$$\frac{\pi}{2\sqrt{3}}$$

$$\frac{1}{2r} \geq \frac{\sqrt{3}}{2} 2r$$

$$r \leq \frac{\sqrt{\frac{2}{\sqrt{3}}}}{2}$$

$$\frac{\pi r^2}{1} = \text{density} \leq \frac{\pi}{\sqrt{12}} \quad //$$

For Lattices... ~~we may consider minimizing the det of the span for a unit lattice.~~

max minimum distance between 2 points... in a unit lattice..

can be small... $\boxed{!}$

but is bounded above...

min $\frac{\varepsilon}{2}$

dist b/w atoms...

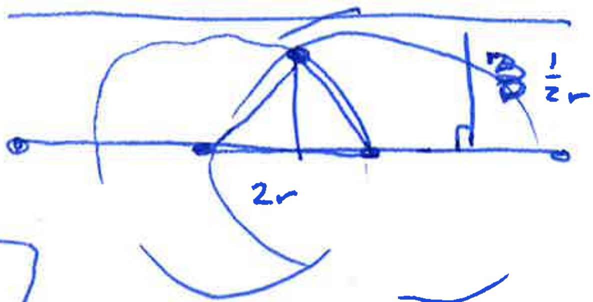
also on ~~$\frac{2}{r}$~~

note

not only orthog

line

$\frac{1}{2r}$ is height of $\parallel$ then



$$\frac{1}{2r} \ge \frac{\sqrt{3}}{2} 2r$$

$\Rightarrow$

$$r \le \sqrt{\frac{2}{\sqrt{3}}}$$

maximum $2r \Rightarrow$ min $\frac{1}{2r}$

~~$\frac{1}{2r} = \frac{\sqrt{3}}{2} 2r = \sqrt{3} r$~~

$$\frac{1}{2r} = \frac{\sqrt{3}}{2} 2r$$

$$\frac{1}{2r} \ge \sqrt{3} r$$

r

1.5

In general, we need some analogous method to partition space.

≡

Classically: Dirichlet Voronoi diagrams are good...

D_i associated with P_i

is

$$D_i = \{x \in \mathbb{R}^2 : d(x, P_i) < d(x, P_j) \ \forall j \neq i\}$$

when P_i is a disk and d = standard distance..

This is also the dist to the center of the circle..

Nice properties... half space decomposition wrt centers, so convex..

partition space... up to a ∂ term.

... picture..

Dual notion: Delaunay Δ-ation.

- non unique ... □ ...

Difficult..

Characterization: circumcircles of Δ's are empty of other points/vertices..

[Its circum center is the Voronoi vertex

Existence:

Consider a collection of 4 points in the plane

~~[illegible]~~

Lift to $z = x^2 + y^2$ paraboloid...

a, b, c, d in plane x-y

lift to $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ on $z = x^2 + y^2$

$d \subset$ circumcircle of a, b, c

$\Leftrightarrow$

$\hat{d}$ is in lower $\frac{1}{2}$ space of $\langle \hat{a}, \hat{b}, \hat{c} \rangle$

A plane $\perp$ to parabola at (p, q)

has the form.

$$z = 2px + 2qy \mp (p^2 + q^2)$$

Shifted ...

$$z = 2px + 2qy + (p^2 + q^2) + h^2$$

$$x^2 + y^2 = \quad ''$$

$$= \quad (x + p)^2 + (y + q)^2 = h^2.$$

IC

So place the $\perp$ to $(p, q, ~~p^2+~~ p^2+q^2)$

at height h^2 is $\overleftrightarrow{p \sim j}$ to x, y

a circle radius h

$\Rightarrow$ lemma . . . i ~~f~~ ...

$\exists$ compact wall $\Rightarrow$ Existence of Rely Δ

1.6

<u>Thre</u>

1.6

Lemma The Largest angle of

$\Delta ABC \in DT$ of a saturated packing.

satisfies:

$$\underbrace{\frac{\pi}{3} \leq \Theta}_{\substack{\text{obvious} \\ \text{by largest} \\ \text{angle} \\ \text{of} \\ \Delta.}} \not\geq \frac{2\pi}{3}$$

Assume $\Theta \geq \frac{2\pi}{3}$. $\Rightarrow$ Circumradius $\Delta ABC > 2$.

if A smallest Angle,

and $\Theta \geq \frac{2\pi}{3} \Rightarrow A \leq \frac{\pi}{6}$

$\Rightarrow \sin(A) \leq \frac{1}{2}$

Circumradius formula.

$$R = \frac{1}{2} \frac{\overline{BC}}{\sin A} \geq \frac{2}{\frac{1}{2}} \cdot \frac{1}{2} = 4$$

$\cdot\times\cdot$

Lemma

$\triangle$ density in a set $\triangle$ DT

$\leq \frac{\pi}{\sqrt{12}}$.) = sharp if equilat.

B Largest Angle $\triangle ABC$.

$\text{area} = \frac{1}{2} \overline{AB} \cdot \overline{CB} \sin B$

$\geq \frac{1}{2} \cdot 2 \cdot 2 \cdot \min_{[\frac{\pi}{3}, \frac{2\pi}{3})} \sin \theta$

$= \frac{1}{2} \cdot 2 \cdot 2 \cdot \frac{\sqrt{3}}{2}$ when $B = \frac{\pi}{3}$

So Area DT $\geq \sqrt{3}$

$\boxed{\text{Theorem}}$

$\Rightarrow$ $\text{density}_{DT} \leq \frac{\pi}{2\sqrt{3}}$ ∎

Density of P

$$= \sum_{\Delta \text{ in } DT} \frac{\text{area } \Delta t \times \text{Density of } DT}{\text{area } DT}$$

$$\leq \frac{\pi}{\sqrt{12}}$$

$\Rightarrow$ any finite union of DT has density $< \frac{\pi}{\sqrt{12}}$.

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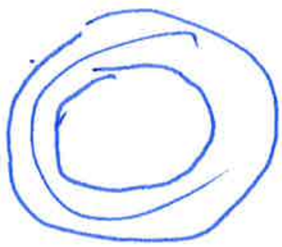
finite version of the density of the Δ is reached here.

So error in the limit is a limit



$\Rightarrow$

$d < \frac{\pi}{\sqrt{12}}$.