Lecture 3 18.4.2016

Last time: Fejes Tóth inequality
Blichfeldt sphere packing Bound.

3.1) Some remarks

- Where is the Fejes Tóth inequality sharp? For n points

$$d \le \arccos\left(\frac{\cot(\omega)-1}{2}\right);$$

$$\omega = \frac{n}{n-2} \cdot \frac{\pi}{6},$$

d angular edge length of the equilateral triangle with area $\frac{4\pi}{2n-4}$ on unit sphere.

So, this should be sharp for an equilateral Δ-hedron...

$n = \underline{3}, 4, 6, 12.$, and perhaps $n \to \infty$

- There are other proofs, longer.

~~Again, There exist e.g. a trivial Blichfeldt gauge for C.~~
~~is given by C.~~

3.2 aside] We saw last time.

$$f_0(x) = \begin{cases} 1 - \frac{1}{2}|x|^2 & x \le \sqrt{2} \\ 0 & x > \sqrt{2} \end{cases}$$

is a Blichfeldt gauge for $\mathbb{B}^n$.

It is non-trivial for our purposes.

that is:

$$\int_{\mathbb{R}^n} f_0(x) \neq \int_{\mathbb{R}^n} \chi_{\mathbb{B}^n}$$

(at least for $n \neq 2$)

$$\int_{\mathbb{R}^n} f_0(x)\, dV = \frac{2^{1+\frac{n}{2}}}{2+n} \operatorname{vol}(\mathbb{B}^n)$$

$\Rightarrow$

$$\frac{\mathrm{Vol}(\mathbb{B}^n \cap \mathrm{Box}(t))}{\mathrm{Vol}(\mathrm{Box}(t))} \le \underline{\text{always}}$$

$$\frac{n+2}{2^{1+\frac{n}{2}}}\left(1+\frac{\sqrt{2}-2}{t}\right)^n$$

$$+ \frac{c\, t^{n-1}}{t^n}$$

$$\longrightarrow \limsup_{t\to\infty}\left(\frac{\mathrm{Vol}\, \mathbb{B}^n \cap \mathrm{Box}(t)}{\mathrm{Vol}(\mathrm{Box}(t))}\right) \le \boxed{\frac{n+2}{2^{1+\frac{n}{2}}}}$$

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note, we could use.

$$\delta = \limsup_{t\to\infty} \frac{\mathrm{Vol}\,\{\mathbb{B}^n : \mathbb{B}^n \subseteq \mathrm{Box}(t)\}}{\mathrm{Vol}\ \mathrm{Box}(t)}$$

3.4) What does this look like?

f_0: Security bounds.

$n=1$ 1.06
1
.88
.75
.61
.5
.39
.31
.24
$n=10$ .19

$$\frac{n+2}{2^{1+\frac{n}{2}}}$$

Lower Bounds:

$\frac{1}{2^n}$ (Saturation)

⋮

Ball (1992?)

$$\frac{2n}{2^n}$$

Vance $n \equiv 0 \bmod 4$

$$\frac{\frac{6}{e}n}{2^n}$$

!!!

Venkatesh (Space) $\frac{e^{-\gamma}}{2} n \log\log(n) / 2^n$

Upper

$\frac{n+2}{2^{\frac{n}{2}+1}}$

⋮

⋮

$2^{-.599n}$ Kabatjanskii ~~Kabatoksky~~ Levenštein

(Cohn, Elkies, Kumar)

- Known good constructions in Low dimensions / Special ones...

need to talk about this : applications at some point ...

Exact Results. (upper)

d = 1 trivial

d = 2 we proved it. (1892... ???) Thue.

d = 3 – Hales, twice → complicated geometric analysis
2000 ish..

d = 8
d = 24 } 1 month ago : LP duality
constructed a special function..

Remember...

$$\mathbb{Z}^n \sim \frac{\pi^{n/2}}{2^n(\cancel{+}\frac{n}{2})!}$$

3.4) Kabatjanskiĭ + Levenštein.

Thm: In high dimensions. for $1 < r < 2$

$$f_d(x) = \begin{cases} M(d, \arccos(1 - 2/r^2))^{-1} & |x| < r \\ 0 & |x| \geq r \end{cases}$$

is a Blichfeldt gauge for $\mathbb{R}^d$

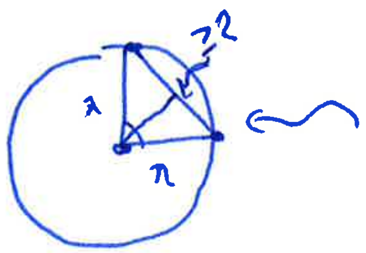
where $M(d, \varphi)$ is the maximum number of points on ~~the (d-1)~~ S^{d-1} with pairwise distance $\geq \varphi$.

or... # pts in $\mathbb{R}^d$ st the angle spanned by any 2 pts is at least φ. (proof to.....)

proof: for any packing of unit balls in $\mathbb{R}^d$, for $r < 2$

There are at most $M(d, \arccos(1 - 2/r^2))$ centers at distance less than r. and can show. to

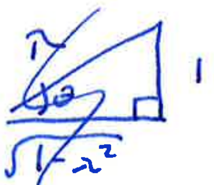
angle at x spanned by segment of length ≥ 2, ends within r from x is at least $\arccos(1 - 2/r^2)$



chord length is. $2 - 2(a \cdot b)$

$\cos\theta = \frac{1}{r}$

$2\arccos\left(\frac{1}{r}\right)$



$\cos\theta = \frac{\sqrt{1-r^2}}{r}$

$A \cdot B = \|A\| \, \|B\| \cos\theta$

$r \cos\theta$

$2 - 2a\cdot b$

$2 - 2r^2 \cos\theta$

$$\left|\frac{2}{r}\right|^2 \le \left|\frac{d}{r}\right|^2 = 2 - 2(u \cdot v)$$

$$= 2 - 2\cos\theta$$

$$\left|\frac{2}{r}\right|^2$$

$$\frac{4}{r^2} \le 2 - 2\cos\theta$$

$$\frac{2}{r^2} \le 1 - \cos\theta$$

$$\boxed{\cos\theta \le 1 - \frac{2}{r^2}}$$

Thm.

$$\delta(B^d) \leq \lambda^{-d} M(d, \arccos(1 - 2/\lambda^2))$$

$1 \leq \lambda \leq 2$.

event.. $\Longrightarrow$

$$\rightsquigarrow \delta(B^d) \leq \sin(\tfrac{\varphi}{2})^d M(d, \varphi)$$

$\frac{\pi}{3} < \varphi \leq \pi$

for d large,
φ ~~small~~... $\leq 63°$..

$$M(d, \varphi) \leq \sin(\varphi/2)^{-d} \, 2^{-d \cdot .599 + \dots}$$

$$\Rightarrow \delta(B) \leq 2^{-d \cdot .599 + o(d)}.$$

3.[illegible])

So:

We defined a Blichfeldt gauge for $\mathbb{R}^n$. Can we extend this?

Abstractly:

Given a (convex) body C in $\mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}$ is a <u>Blichfeldt gauge for C</u> if,

for any collection of isometries $\{\varphi_i\}_{i=1}^{\infty}$ of $\mathbb{R}^n$ s.t. $\{\varphi_i C\}_{i=1}^{\infty}$ is a packing,

$$\sum_{i=1}^{\infty} f(\varphi_i^{-1} x) \leq 1 \quad \forall \; x \in \mathbb{R}^n$$

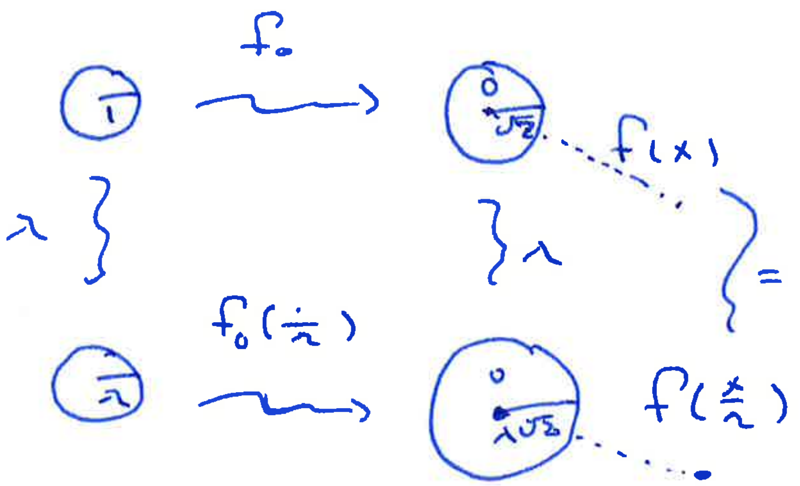
- Remark... could add non-negativity condition... really want mass of f: $I(f) \geq I(\chi_C)$
- Remark. the index set could also be finite...
- Remark... f could be much stronger than we allow... [illegible] fast..
- Exercise *** (Think about such functions distributionally?)

There exists a blichfeldt gauge for C.

eg. χ_C.

- If $f(x)$ is a Blichfeldt gauge for C,

  $f(x/\lambda)$ is a Blichfeldt gauge for λC, the λ scaling of C.



For a body C, define its insphere radius to be $r(C)$, the radius of the largest sphere contained in C

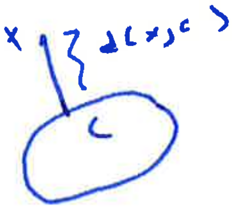
$\boxed{\bigcirc}$

For $0 \le \gamma \le r(C)$ define C_γ to be the inner parallel body at depth γ (42)

The set of points

$$C_\gamma := \{ x \in C \text{ st } x + \gamma B^d \subset C \}$$

we can define the straight line distance to a body by $d(x, C)$



Then... if f is a radial Blichfeldt gauge for B^n,

ie $f(x) = F(|x|)$

Then $g(x) = F\left(\frac{d(x, C_\gamma)}{\gamma} \right)$

define.

Claim... $g(r)$ is a Blichfeldt gauge for C.

This is fairly easy to see by picture...

~~Consider a point $x \in \mathbb{R}^n$.~~

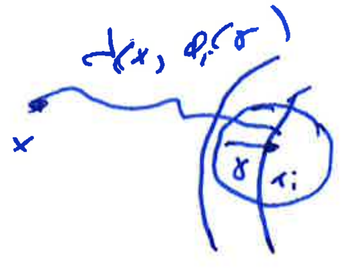
We would like to show that

$$\sum_{i=1}^{\infty} g(\varphi_i^{-1} x) \leq 1 \quad \text{for all packings of } C \ \{\varphi_i C\}_{i=1}^{\infty}.$$

$$\forall x \in \mathbb{R}^n$$

Consider a point $x \in \mathbb{R}^n$
and a pt x_i in $\varphi_i C_\gamma$

at distance
$d(x, \varphi_i C_\gamma)$ from x.



Then $\gamma B^n_{x_i}$ is in C

$\Rightarrow$ $\{\varphi_i \gamma B^n_{x_i}$ is a $\}$.

packing

(Since C is convex, then all
well defined..)

Since

$\Rightarrow$ $F\left(\frac{x}{\gamma}\right)$ is a Blackwell dependent? function

γB^n

$\Rightarrow$

and C is convex.. and x is

and distance.. $d(x, C\cdot\gamma)$

$$\Rightarrow \sum B \, F\left(\frac{d(x, C_\gamma)}{\gamma}\right)$$

$$\leq 1 \,.$$

As with Blichfeldt guys and other approaches... is there valid?

if we have a convex body like an

$A + B^n$, it could be..., provided there is enough high dimensional moves...

$$f(x) = \begin{cases} f_0 & |x| > 1 \\ 1 - f_0(2 - x) & |x| \le 1 \end{cases}$$

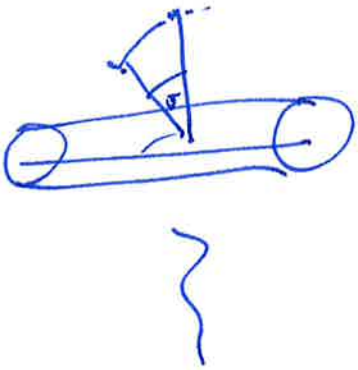
$$\rightsquigarrow \quad \delta \le \left(\frac{2}{d+2} \sqrt{2}^{\,d} (1 + b_d) \right)^{-1}$$

$$b_d = \frac{1}{(\sqrt{2})^d (d+1)} \cdot -(\sqrt{2} - 1)^{d+1} \left(1 + \frac{\sqrt{2}}{d+1}\right)$$

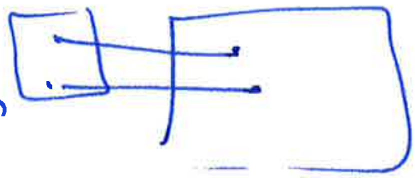
$\delta(f...)$
f Bd...

1	f_0	1.06
.94		1
.84		.88
.72		.75
.60		.61
.49		.5
.39		.39
.31		.31
.24		.24
.18		.19

So this works for things with a lot of "2d" mass...
or bigger



vs



.94 + Error

1 + Error...

~~tan~~

never doubt that
there is no
1d blackbody
Energy...

ℓ capped.

$$\overline{\quad \ell \quad} \times B^n$$

$$I(g) = \cancel{b} \cdot \frac{I_{d-1} f}{V}$$

$$\ell \cdot \frac{\int_{d-1}(f)}{V(B^{d-1})} \cdot VB^{d-1} + \frac{\int_d f}{VB^d} \, V_d \, b.$$

$$V(\ell \text{ capped})$$

$$= \cancel{b}\,\ell V(B^{d-1}) + VB^d.$$

$$=)\cdot$$

$$\delta(\ell \text{ cap}) \leq \frac{\ell(VB^{d-1}) + VB^d)}{\cancel{k}\, I_{d-1} f + I_d f.}$$

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end errors for cylinder..
