Lecture 4) The classification of (crystallographic) root systems
(and 5?)

4.1) We will examine root lattices (crystallographic):
One motivation is to find nice constructions, in low dimensions;
Another is to show that ~~there are not~~ such simple constructions don't really extend to high dimensions in a good way.

The best known packings up through $\mathbb{R}^8$ are root lattices.

Covers $\frac{1}{2}$ of K

Classification of (complex) (semi) simple Lie algebras (~ complex Lie groups). $\frac{1}{2}$

root systems

flows

to get another

A_1, A_2, A_3, E_8 optimal as gen packings.

A Lattice is a discrete (+) subgroup of $\mathbb{R}^n$. For our purposes... it will be co-compact or have full rank, it will have a $\mathbb{Z}$ basis $\{a_1 \ldots a_n\}$ that spans as an $\mathbb{R}$ basis.

e.g. $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^2$

Can have different bases, but since such bases differ by an integer change of basis T (with integral inverse)

$$a_i' = \sum_{i=1}^{n} k_i a_i \qquad k_i \in \mathbb{Z}$$

$$\Rightarrow \det\{k_i\} = \pm 1$$

So the volume of $\mathbb{R}^n / \Lambda$

$= |\det \Lambda|$

$= \mathrm{Vol\ span}\{a_1 \dots a_n\}$

is a clear invariant of Λ.

---

so we can normalize (to SL_n, compact)

Then there is a clearly upper bound on the length of the shortest vector on an lattice of $|\det \Lambda| = 1$ so a bound on

The largest volume sphere that can be ~~placed~~ fit with at the Λ q.

$\varphi_i \in \Lambda$ for a packing

$\{ \varphi_i, B(\lambda_{min} \frac{\Lambda}{2}) \}$

λ_{min} upper $\frac{\Lambda}{2}$

---

so... what is the optimal lattice?

Optimal lattice packings... ~~Still open~~. Blichfeldt 1935 ... up to $n = 8$.

$\left\{ \begin{matrix} A_1\ A_2\ A_3\ D_4\ D_5 \\ E_6\ E_7\ E_8 \end{matrix} \right\}$

These are root lattices.

4.2) Good candidates in low dimensions

A <u>root lattice</u> is an <u>integral lattice</u> generated by <u>roots</u>.

$$\boxed{\begin{array}{c}\text{integral lattice } \Lambda \\ \iff \\ \langle x, y\rangle \in \mathbb{Z} \\ \forall\, x, y \in \Lambda\end{array}}$$

roots: elements of a ~~roots~~ (<u>root system</u>)

$\Delta \subset \mathbb{R}^n \setminus \{0\}$ ~~with~~ (satisfying)

1) Δ finite, span $\mathbb{R}^n$

2) $\alpha \in \Delta \Rightarrow n\alpha \in \Delta \iff n = \pm 1$ (reduced)

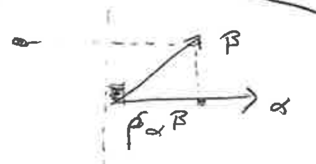
3) Δ invariant under $\perp$ reflection to any $\alpha \in \Delta$

$$\mathrm{proj}_\alpha \beta = \alpha \frac{\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle}$$

$$s_\alpha \beta := \beta - 2\,\mathrm{proj}_\alpha \beta \in \Delta$$

(Spiegel)

$\Delta \ni s_\alpha \beta$



4) Crystallographic

$$n_{\beta\alpha} := \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}.$$

property 4 is the restriction

that the projection of β onto

α is an integer or $\frac{1}{2}$ integer

multiple of α, but it

really forces. $\Delta \rightsquigarrow \perp$

encode

Lattice ...

Ex. $\boxed{\text{Try building up some other root systems, non-crystallographic}}$

$\longrightarrow$

Notice: 4) is strong...

$$n_{\beta\alpha} = 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \frac{2|\beta|}{|\alpha|}\cos\theta$$

since $n_{\beta\alpha}$ and $n_{\alpha,\beta} \in \mathbb{Z}$

$\Rightarrow$

$$\Rightarrow \boxed{n_{\beta\alpha} n_{\alpha\beta} = 4\cos^2\theta \in \mathbb{Z}}$$

$$\Rightarrow 4\cos^2\theta \in \{0, 1, 2, 3, 4\}$$

and for

$$4\cos^2\theta = 4$$

$\Rightarrow$

$\alpha = \beta$ or

$\alpha = -\beta$

$4\cos^2\theta$	$n_{\beta\gamma}$	$n_{\gamma\beta}$	$\lvert\alpha\rvert / \lvert\beta\rvert$	$\cos\theta$	θ
3	+1	+3	$\sqrt{3}$	$+\sqrt{3}/2$	$\pi/6$
3	−1	−3	$\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$5\pi/6$
2	+1	+2	$\sqrt{2}$	$+\frac{\sqrt{2}}{2}$	$\pi/4$
2	−1	−2	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$3\pi/4$
1	+1	+1	1	$\frac{1}{2}$	$\pi/3$
1	−1	−1	1	$-\frac{1}{2}$	$2\pi/3$
0	0	0	(—)	0	$\frac{\pi}{2}$

$|\gamma| \geq |\beta|$

↑

Table assum

reducible

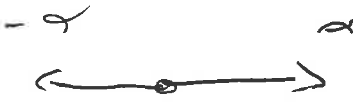
A root system is (decomposible)

if $\Delta = \Delta_1 \cup \Delta_2$ st

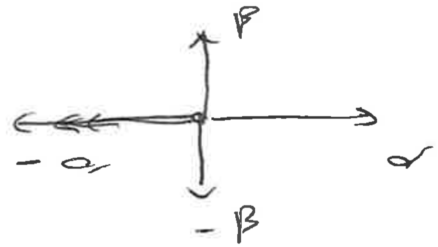
$\forall \; \alpha_1 \in \Delta_1, \alpha_2 \in \Delta_2,$

$\langle \alpha_1, \alpha_2 \rangle = 0$. ie

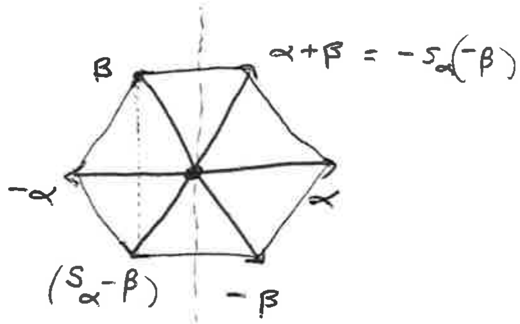
all components of 1 are orthogonal to all compts of another. else call it irreducible. (indecomposible.)

A_1 

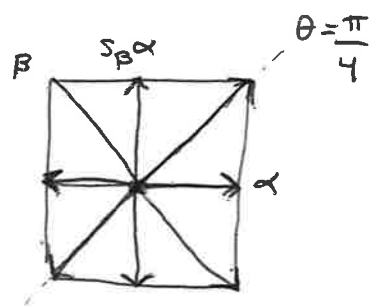
$n = 1$

$A_1 \times A_1$ (decomposible) 

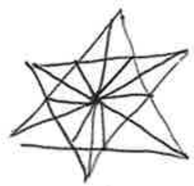
$\theta = \frac{\pi}{2}$

A_2 

$\theta = \frac{\pi}{3}$

B_2 

$\theta = \frac{\pi}{4}$

G_2 
$\theta = \frac{\pi}{6}$

(note: Δ is not a basis!)

(4.3) Classification of root systems

preliminaries

From a root system Δ

we can choose a (non-unique)

subset, the simple roots.

- For each $\alpha \in \Delta$, ~~there is a unique $\perp$ hyperplane its~~

consider $\alpha^\perp$, its (linear) orthogonal complement

since $|\Delta|$ is finite, $\Rightarrow \exists\, d$

st $\forall\, \alpha \in \Delta$, $\langle \alpha, d \rangle \neq 0$.

$\Rightarrow \exists$ partition Δ into

positive $\Delta_d^+ = \{\alpha \in \Delta : (\alpha, d) > 0$

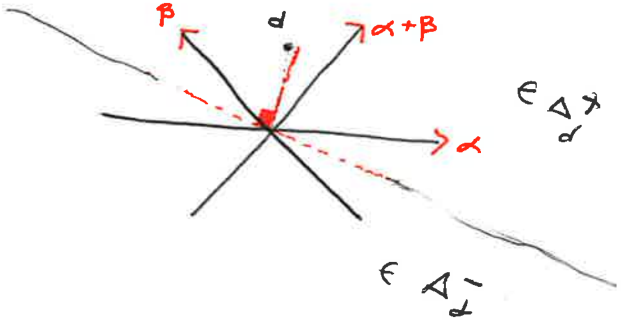
negative $\Delta_d^- = \{\alpha \in \Delta : \langle \alpha, d \rangle < 0$

A root $\alpha \in \Delta_d^+$

is simple if it is

not the sum of

2 other positive roots.



A set of simple roots is a fundamental system of Δ.

- The choice of δ affects the fundamental system. but such systems are equivalent under the actions of reflections ~~through~~ group $\langle S_\alpha \rangle_{\alpha \in \Delta}$.

$=$

a fundamental system of Δ is an $\mathbb{R}$-basis for $\mathbb{R}^n$

Sketch: (Lin ind)

$$\sum_{i \in I} c_i \gamma_i = 0, \quad \gamma_i \in \Phi_s$$

partition into $c_i > 0 = (+)$
are zero $\quad c_i < 0 = (-)$

$$\Rightarrow (+)^2 = (+)(-)$$

$$= \underbrace{|c_+||c_-|\langle \alpha \cdot \beta \rangle}_{} \leq 0$$

(span)

$\gamma \perp$ to all simple roots

Then by arbitrary.
by $\langle S \rangle$
$\gamma \perp$ to all roots.

Δ can be reconstructed from a fundamental system via reflection

$\boxed{\text{picture}}$.

Sketch. ~~orbit d. to another~~

Consider <u>Weyl chamber</u> of f_S (finite spanning set.)

$$\{ x \in \mathbb{R}^n \text{ st } \langle x, f_S \rangle > 0 \}$$

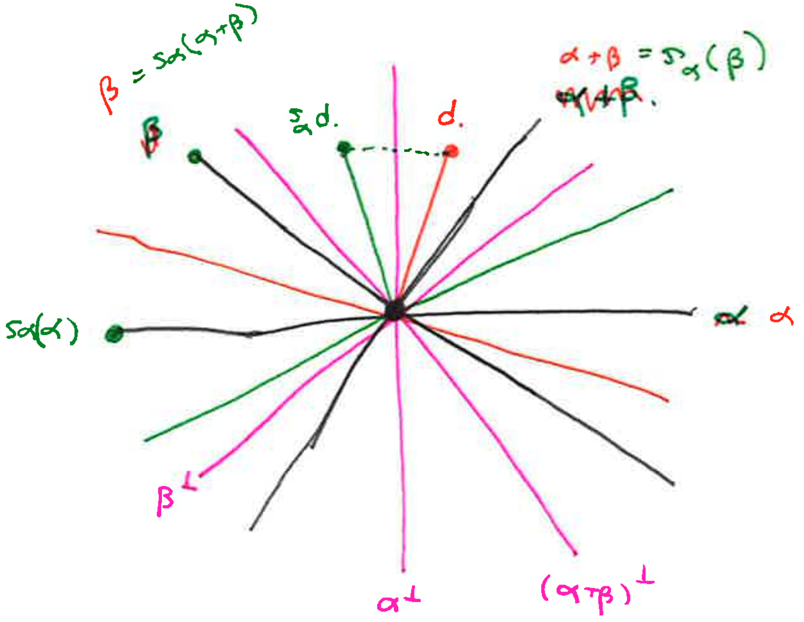
~~then try to get to another chamber.~~

Then $d \in$ Weyl cham doesn't change the f_S.

any other f_S is based on d' in another chamber. (finite set of chambers)

path $d \to d'$ then faces exists

$\to$ faces $d \to d'$ (chambers).



$\beta = s_\alpha(\alpha+\beta)$
$\alpha+\beta = s_\alpha(\beta)$
β
$s_\alpha d.$
$d.$
$s_\alpha(\alpha)$
α
α
$\beta^\perp$
$\alpha^\perp$
$(\alpha+\beta)^\perp$

If α, β not $\|$, $(\alpha, \beta) > 0 \Rightarrow$

$\alpha - \beta \in \Delta$.

$=$

$$n_{\alpha\beta} := \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} > 0$$

$\Rightarrow$ $n_{\alpha\beta}$ or $n_{\beta\alpha} = 1$

$$n_{\alpha\beta} = 1 \Rightarrow \boxed{\begin{aligned} s_\beta(\alpha) &= \alpha - 2\beta\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \\ &= \alpha - \beta n_{\alpha\beta} = \alpha - \beta \in \Delta. \end{aligned}}$$

Similarly.

$$m_{\beta\alpha} = 1 \Rightarrow s_{\cancel{\alpha}}(-\beta) = \alpha - \beta$$

If α, β distinct simple roots

$\langle \alpha, \beta \rangle \leq 0$

If $\langle \alpha, \beta \rangle > 0$

Then $\alpha - \beta = \gamma \in \Delta$

$\Rightarrow \gamma$ or $-\gamma \in \Delta^+$

and

$\alpha = \beta + \gamma$ or

$\beta = \alpha + (-\gamma)$ $\times$

{fs} simple roots decomposable $\iff$ root system decomposable.

— similar argument —

4.4) Classification via Coxeter graphs

Simple roots $\leftrightarrow$ root systems

classify

elements of a fundamental system α, β are $\perp$ or obtuse angle.

$$\Rightarrow \theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}.$$

That is:

$$n_{\alpha\beta} n_{\beta\alpha} = 4\cos^2\theta \in \{0, 1, 2, 3\}$$

The Coxeter graph of Δ has a vertex for each simple root and an edge ~~of~~ of ~~the~~ weight $n_{\alpha\beta} \cdot n_{\beta\alpha}$ between α and β //

Well defined from the Weyl chamber up to, since simple roots $\to$ simple roots

$f_c \longrightarrow f_{c'}$

by reflections....

$\langle s_\alpha \rangle$

The Dynkin diagram is the Coxeter graph with arrows on o=o and o≡o edges pointing to the shorter vector.

(Lengths are determined by angles if possible of irreducible system.)

Ex.

$A_1 \longrightarrow$ o

$A_1 \times A_1 \longrightarrow$ o o

on relabel. scale! other.

$A_2 \longrightarrow$ o—o

$B_2 \longrightarrow$ o⇒o

$G_2 \longrightarrow$ o⭆o

End Lecture 4)

Lecture 5

Classification of Crystallographic Root systems. (ctd.)

Recall: A root system $\Delta \subset \mathbb{R}^n \setminus \{0\}$ ~~is~~ satisfies.

1) Δ finite, spans $\mathbb{R}^n$

2) $\alpha \in \Delta \Rightarrow (n\alpha \in \Delta \Leftrightarrow n = \pm 1)$

3) Δ invariant under $\perp_\alpha$ reflection.

$(s_\alpha(\beta)) \in \Delta \quad \forall \alpha, \beta \in \Delta$

4) $n_{\beta\alpha} := \dfrac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$

$\equiv$

$\longrightarrow 4\cos^2 \theta_{\alpha\beta} \in \mathbb{Z}$, really $\in \{0, 1, 2, 3, 4\}$

root lattices

$\rightarrow$ root systems

$\longleftrightarrow$ Last time.

simple roots.

$\longleftrightarrow$ Today

Dynkin Diagrams

from 4.4)

Coxeter Graph

Vertex for each simple root, edge of weight $n_{\alpha\beta} \cdot n_{\beta\alpha}$ between α and β.

Dynkin Diagram

Coxeter graph with arrows on multiedges pointing at ~~larger~~ (~~longer~~) shorter root.

Lecture 5) Classification of Coxeter graphs.

(ignore lengths)

an independent ~~set~~ set of n unit vectors. $\{v_1 \dots v_n\}$ spanning $\mathbb{R}^n$ is admissible if $\forall$ $i \neq j$, $\langle v_i, v_j \rangle \leq 0$

and

$$4 \langle v_i, v_j \rangle^2 = 4 \cos^2 \theta_{ij} \in \{0, 1, 2, 3\}$$

$$\boxed{\text{A normalized set of simple roots is admissible.}}$$

(note, angles are the defining feature)

An admissible diagram is the Coxeter graph of an admissible set.

The simple roots of an irreducible set Δ are not all orthogonal

$\Rightarrow$

Coxeter graph is connected.

so only need to classify

Connected admissible diagrams

Thm: The Dynkin Diagram of an irreducible root system is of the ~~[illegible]~~ type.

n vertices.

A_n 

$n \geq 1$

B_n 

$n \geq 2$

C_n 

$n \geq 3$

D_n 

($n \geq 4$

(each of these anyhow)

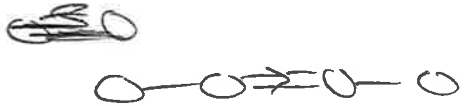
infinite families

Exceptional of type

E_6 

E_7 

E_8 

~~E_2~~ F_4 

G_2 

It will suffice to classify admissible diagrams.

B definition, subsets of vectors. satisfy the admissibility condition inside their span (could be disconnected)

Lemma

A connected admissible diagram is a tree.

Consider $v = \sum_{i=1}^{n} v_i$, $\{v_i\}$ admissible.

v_i lin indp $\Rightarrow$ $v \neq 0$

$0 <$

$\Rightarrow \langle v, v \rangle = \sum \langle v_i, v_i \rangle$

$+ \sum_{i<j} 2 \langle v_i, v_j \rangle$

$= n + \sum_{i<j} 2 \langle v_i, v_j \rangle$

if v_i is connected to v_j in the Coxeter graph, then the table

$\Rightarrow$

$2 \langle v_i, v_j \rangle$

$\in \{-1, -\sqrt{2}, -\sqrt{3}\}$

$\Rightarrow \sum_{i<j} 2\langle v_i, v_j\rangle$ has

at most $n-1$ terms.

not equal to 0

$\Rightarrow$ at most $n-1$ connected vertices.

but by assumption, it is connected $\Rightarrow$ exactly $n-1$ pairs of connected vertices $\Rightarrow$ tree

(maybe with multiple edges but not too many)

Lemma ~~Each~~ of each vertex degree is at most 3, with multiplicity.

Fix vertex v, connected to vertices $\{v_1 \dots v_k\}$.

Tree $\Rightarrow \langle v_i, v_j\rangle = 0$ for $i \neq j$

$\Rightarrow \{v_1 \dots v_k\}$ orthonormal.

also, $\{\{v_1 \dots v_k\}, v\}$ are lin ind (simple roots) Lemma

$$\Rightarrow \frac{\text{proj}_{\langle v_1 \dots v_k\rangle^\perp}(v)}{\| \quad " \quad \|} = v_0 \neq 0$$

and

$\{v_0, \ldots v_k\}$

orthonormal at.

$\Rightarrow$

$$v = \sum_{i=0}^{k} \langle v, v_i \rangle v_i$$

<u>and</u>

$$\langle v, v \rangle = \sum_{i=0}^{k} \langle v, v_i \rangle^2 = 1$$

since

$\langle v, v_0 \rangle^2 \neq 0 \quad \Rightarrow$

$$\sum_{i=1}^{k} 4 \langle v, v_i \rangle^2 < 4$$

but

$4 \cos^2 \theta$

$= 4 \langle v, v_i \rangle^2 = \#$ edges.

$=$ ~~[illegible]~~

from

v on

to v_i

$\Rightarrow$ degree of v ~~# of edges~~ w/

multiplicity is at

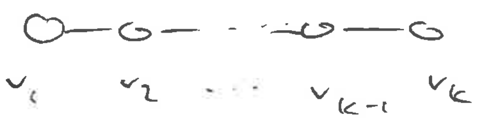
most 3.

Cor. OEO is the only admissible diagram with a triple edge.

Consider only double and single edges from now on.

~~Lemma (Simple Chains) Collapse.~~

Def: A simple chain is a collection of vertices connected by single edges.



Lemma (Simple Chain Collapse.)

A simple chain representing $\{v_1 \dots v_k\}$ can be replaced by $v = \sum_{i=1}^{k} v_i$ yielding an admissible diagram.

WTS... v unit vector and the collapsed diagram is admissible.

$$\langle v, v\rangle = k + \sum_{i<j} 2\langle v_i, v_j\rangle$$

and $\langle v_i, v_j\rangle = 0 \quad \forall i \neq j$

except $j = i+1$

(recall)

single edge $\Rightarrow$ $2\langle v_i, v_{i+1}\rangle = -1$

$$\Rightarrow \cancel{k}\ \langle v, v\rangle = k-(k-1)$$

$$= \underline{1}$$

~~recall admissible~~

Consider u not represented in the chain.

it connects to only $\underline{1}$ vertex in the chain (by tree), say v_j.

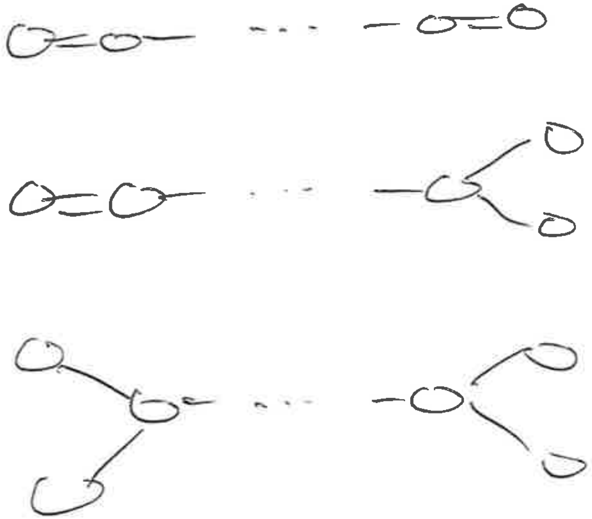
$$\langle u, v\rangle = \sum_{i=1}^{k} \langle u, v_i\rangle$$

$$= \langle u, v_j\rangle$$

$\Rightarrow$ the angle ~~with~~ $\theta_{u,v}$ and θ_{u,v_j} ~~are the~~ ~~the~~...

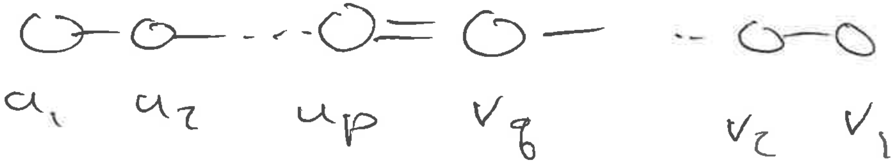
are the same $\Rightarrow$ admissible.

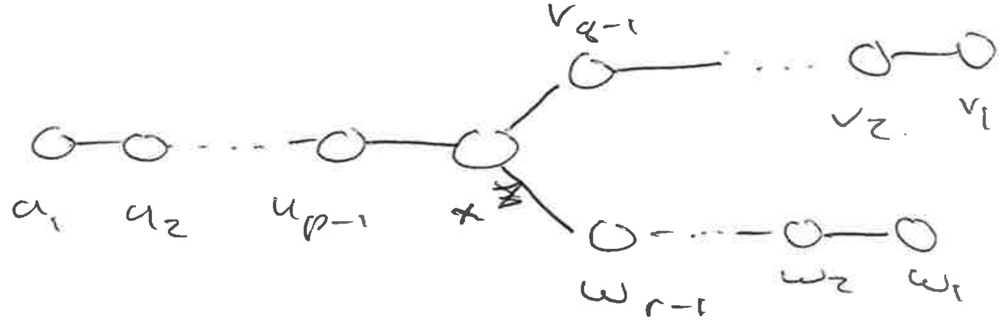
So Coxeter graphs of type:



forbidden.

Can contain at most
1 branch
XOR
1 double edge





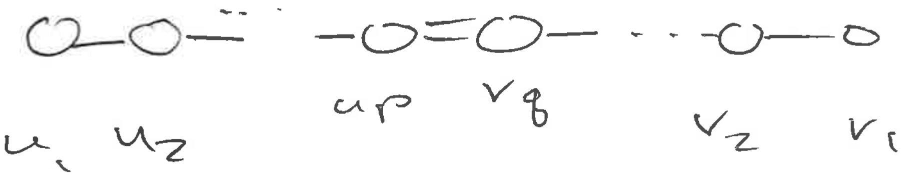
The simple chains are

or $\longrightarrow A_n$.

B_n

C_n

F_4



$\circ$ $\quad u := \sum_{i=1}^{p} i u_i$

$\circ$ $\quad 2\langle u_i, u_{i+1}\rangle = -1 \quad \forall\, 1 \le i \le p-1$

$$\langle u, u\rangle = \sum_{i=1}^{p} i^2 \langle u_i, u_i\rangle + \sum_{i<j} 2ij \langle u_i, u_j\rangle$$

$$= \sum_{i=1}^{p} i^2 + \sum_{i=1}^{p-1} (i(i+1)) \cdot \underbrace{2\langle u_i, u_{i+1}\rangle}_{-1}$$

$$= \cancel{\cdots}$$

$$\sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1)$$

$$= p^2 - \sum_{i=1}^{p-1} i$$

$$= p(p+1)/2$$

Similarly

$$\langle v, v \rangle = \frac{q(q+1)}{2}$$

---

so $\langle u, v \rangle = pq \langle u_p, v_q \rangle$

also

~~by edges~~ ~~note~~

~~since only $v=0$ is not~~

~~orthogonal~~ relations

since $v=0$ is the only

non orthogonal relation,

and $\quad 4\langle u_p, v_q \rangle^2 = 2$

$\Rightarrow$

$$\langle u, v \rangle^2 = \frac{p^2 q^2}{2}$$

and

since u, v not $\parallel$

$$\langle u, v \rangle^2 \underset{\neq}{\leq} \langle u, u \rangle \langle v, v \rangle$$

$\Rightarrow$

$$\frac{p^2 q^2}{2} < \frac{p(p+1)}{2} \frac{q(q+1)}{2}$$

$p, q \in \mathbb{Z}$

So

$$2pq < (p+1)(q+1)$$

$$\Rightarrow \boxed{(p-1)(q-1) < 2}$$

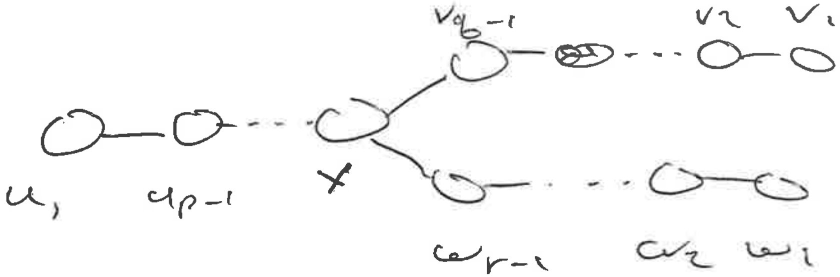
$$\Rightarrow \quad p = q = 2 \qquad \boxed{F_4}$$

or

$$p = 1 \quad q = (\text{—}) \qquad \boxed{B_n, C_n}$$

Finally

D_n



E_6

Same as previous

E_7

$$u = \sum_{i=1}^{p-1} i\, u_i$$

E_8

note. u, v, w are mutually orthogonal vectors.

---

x is also not in

$\langle u, v, w \rangle$

$$1 = \langle x, x \rangle \geq \underbrace{\langle e, \tfrac{u}{\|u\|} \rangle^2 + \langle x, \tfrac{v}{\|v\|} \rangle^2 + \langle x, \tfrac{w}{\|w\|} \rangle^2}$$

projection length in $\langle u, v, w \rangle$.

also

$\langle x, u_i \rangle^2 = 0$ unless

$i = p-1$ and

$4 \langle x, u_{p-1} \rangle^2 = 1$

(single edge)

Then. $\langle x, u \rangle^2 = \sum_{i=1}^{p-1} i^2 \langle e, u_i \rangle$

$= (p-1)^2 \langle x, u_{p-1} \rangle^2 = \frac{(p-1)^2}{4}$

and $\langle u, u \rangle = \frac{p(p-1)}{2}$ (p → p-1 forever lost)

$\Rightarrow$

$\langle x, \frac{u}{\|u\|} \rangle^2 = \frac{(p-1)^2}{4} \cdot \frac{2}{p(p-1)}$

$= (1 - \frac{1}{p})/2$

Then

$$2 > \left(1-\frac{1}{p}\right)+\left(1-\frac{1}{q}\right)+\left(1-\frac{1}{r}\right)$$

$$\Rightarrow \frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 1$$ and we assume $p, q, r \geq 2$

---

assume. $p \geq q \geq r \geq 1$, integers.

$\Rightarrow r = 2$

Then $q = 2 \Rightarrow p = (\quad)$

$q = 3 \Rightarrow \frac{1}{q}+\frac{1}{r} = \frac{5}{6}$

$\Rightarrow 3 \leq p < 6$

$q = 4 \Rightarrow \emptyset$

Exceptional of

A_n
B_n
C_n
D_n
$E_{6\ 7\ 8}$
F_4
G_2

$\boxed{\text{Construct}}$

next

time

cute.

en.

6) Root systems $\to$ Root Lattices.

6.1 Construction of crystallographic root systems.

A_n: simple roots are all orthogonal or at $2\pi/3$.

Consider $x \in \mathbb{Z}^{n+1}$ such that

$\sum_{i=1}^{n+1} x_i = 0$. This is an

n-dim subspace S of $\mathbb{Z}^{n+1}$

Let $\Delta = \{ x \in S : \|x\| = \sqrt{2} \}$

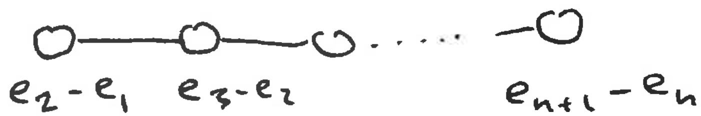
That is, all but 2 coordinates are 0, 1 coordinate is +1, 1 coordinate is -1

$\{1, -1, 0, \dots, 0\}$

$\Rightarrow |\Delta| = n(n+1)$

Choice of simple roots

$\alpha_i = e_i - e_{i+1}$



This looks like.

$$\begin{bmatrix} -1, 1, 0 \cdots\cdots & \\ 0, -1, 1, \cdots\cdots & \\ & -1, 1 \end{bmatrix} \longrightarrow$$

check. Dynkin Diagram.

$$\frac{4 \cdot \langle \alpha_i, \alpha_b \rangle^2}{4} \in \{1, 0\}$$

note: These roots generate the full subspace.

$$x \in \mathbb{Z}^{n+1} \text{ s.t. } \sum_{i=1}^{n+1} x_i = 0$$

So A_n root lattice is all of S.

Maybe not so useful for visualizing?

A_2 

$*-1$

Simple $\begin{cases} (-1, +1, 0) \\ (0, -1, 1) \end{cases}$

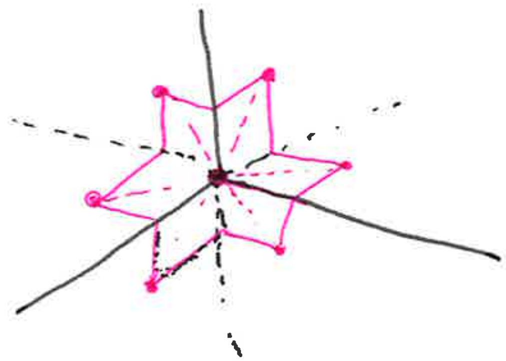
$+$ ———————

$(-1 \ 0 \ 1)$

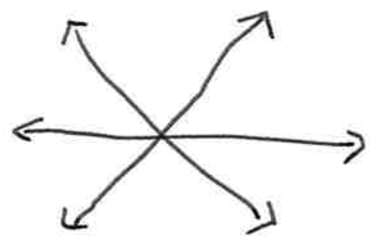
$(1, -1, 0)$

$(0, 1, -1)$

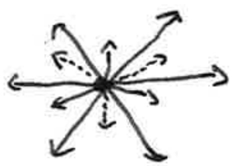
$(1, 0, -1)$



?



A_3)



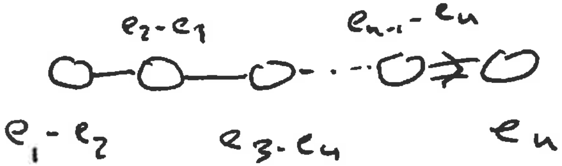
FCC system.

B_n: Δ is all integer vectors $x \in \mathbb{Z}^n$ with norm 1 or norm $\sqrt{2}$

$$\#|\Delta| = 2n + 4\binom{n}{2} = 2n^2$$

simple roots

$\alpha_i = e_i - e_{i+1}$, short root $\alpha_n = e_n$



$$\begin{matrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & & \\ & & & 1 & -1 \\ & & & & 1 \end{matrix}$$

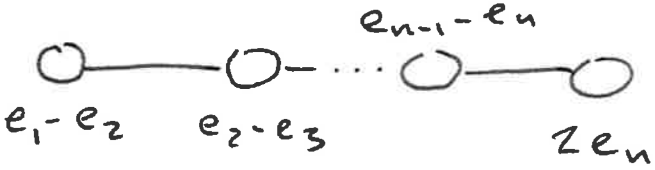
Sadly, from this construction, it seems the associated root lattice is all of $\mathbb{Z}^n$

C_n: Δ is all integer vectors $x \in \mathbb{Z}^n$ of the form $2 \cdot x$, x of length 1, integer, or of length $\sqrt{2}$.

$$|\Delta| = 2n + 4\binom{n}{2} = 2n^2$$

simple roots.

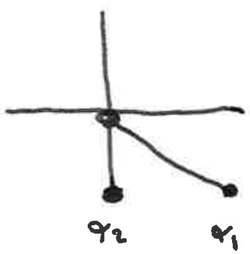
$\alpha_i = e_i - e_{i+1}$ long root $\alpha_n = 2e_n$



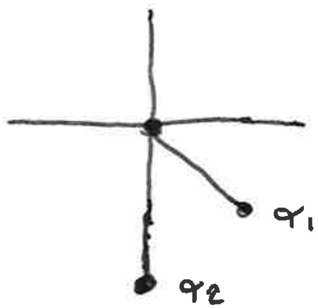
$1, -1$
$\quad 1, -1$
$\quad\quad \ddots$
$\quad\quad\quad 1, -1$
$\quad\quad\quad\quad 2$

So the C_n root Lattice is all integer vectors coordinate sum <u>even</u>

B_2



C_2



D_n $\quad \Delta$ = all integers vectors of length $\sqrt{2}$.

$x \in \mathbb{Z}^n$

$$|\Delta| = 2\binom{n}{2} = 2(n(n-1))$$

Simple roots

$\alpha_i = e_i - e_{i+1}$ and $\alpha_n = e_n + e_{n-1}$

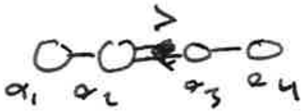
$1 \; -1$
$\quad 1 \; -1$
$\quad\quad \ddots$
$\quad\quad\quad 1 \; -1$
$\quad\quad\quad\quad 1 \; -1$
$\quad\quad\quad\quad 1 \quad 1$

This also generates the lattice $\{ x \in \mathbb{Z}^4$ st component sums are even. $\}$

I can't draw D_4, sadly.

F_4 Δ = set of vectors ~~*~~ $x \in \mathbb{R}^4$ of length 1 or $\sqrt{2}$ with $2x$ coordinates all even or all odd integers

Simple roots.



$$\left.\begin{array}{ll} \alpha_1 & \{1 \ -1 \ 0 \ 0\} \\ \alpha_2 & \{0 \ 1 \ -1 \ 0\} \\ \alpha_3 & \{0 \ 0 \ 1 \ 0\} \\ \alpha_4 & -\frac{1}{2}\{1, 1, 1, 1\} \end{array}\right\} \text{check by hand.}$$

G_2 We already drew this one.

F_4 root lattice.

all coordinates are integers or all coordinates are $\frac{1}{2}$ integers

no mixed

G_2 root lattice $\longleftrightarrow$ A_2 root lattice.

E_8 root system.

6.2) $|\Delta| = 240$

roots of the form

$\pm e_i \pm e_j$ and

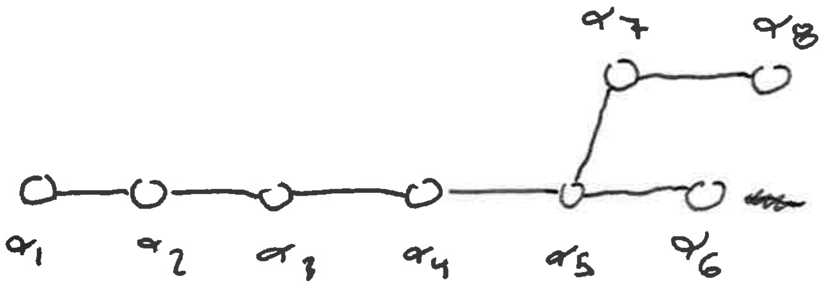
$\frac{1}{2}\{\pm e_1 \pm \dots \pm e_8\}$

$$\begin{matrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ & & 1 & -1 & 0 & 0 & 0 & 0 \\ & & & 1 & -1 & 0 & 0 & 0 \\ & & & & 1 & -1 & 0 & 0 \\ & & & & & 1 & -1 & 0 \\ & & & & & 1 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{matrix}$$

$\alpha_i = e_i - e_{i+1}$

$\alpha_7 = e_7 + e_6$

$\alpha_8 = -\frac{1}{2}(1, \dots, 1)$



This generates the Lattice

1) where coordinates are all integers or $\frac{1}{2}$ integers.

2) sum of coordinates is even.

Clearly related to D_8

Consider the "deep holes" of the D_n lattice ie, the $\frac{1}{2}$ integer points...

$$\| \tfrac{1}{2}(\underbrace{1, \dots 1}_{8}) \| = \sqrt{2}$$

and a typical lattice vector.

$$\| (1, 1, 0 \dots) \| = \sqrt{2}.$$

and similarly.

$$\| (1, 1, 0 \dots) - \tfrac{1}{2}(1, 1, \dots 1) \| = \sqrt{2}$$

So if we have D_8, there is another copy that fits into D_8 !

E_7, E_6. let $v = \{1, 1, \dots 1\}$

$$E_7 = E_8 \cap v^{\perp}$$

$$w = -e_7 - e_8$$

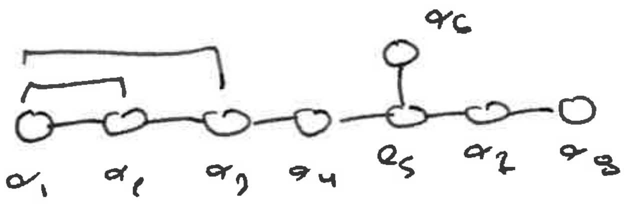
$$E_6 = E_8 \cap (w, v)^{\perp}$$

These are not so satisfactory!

but perhaps are easier to see ~~as~~ constructed ~~[illegible]~~ constructu gu by collapsing the simple choices.

$\alpha_1, \alpha_2 \to \alpha_1 + \alpha_2$

$\alpha_1 + \alpha_2, \alpha_3 \to \alpha_1 + \alpha_2 + \alpha_3$.



This continues to

D_5

D_4

A_3

A_2

A_1

Other constructions of E_8

Exercise ?

6.3 Before we had.

$A_1, A_2, A_3, D_4, D_5, E_6, E_7, E_8$

Best lattice packings, best known packings.

Some how these all are related to E_8.

Dimension 9 is, as far as I know still unknown.

Best constructions for lattices come from an "inductive" process.

given even integers in $\mathbb{R}^1$ Λ_1

The n dimensional laminated Lattice Λ_n, maximum density over all lattices of shortest vector $|x|=2$ containing Λ_{n-1} as a sub lattice.

So...

A_1 A_2 A_3 E_8

$\boxed{\Lambda_9}$

by gluing $\langle \Lambda_{n-1}, g \rangle$

ie

Laminated Lattices

$\Lambda_1 \dots \Lambda_8$

coincide with the root lattices.

Unusual...

cts

family of lattices

that contain Λ_9

same density...

6.4

Dimension 10

a non-lattice packing beats the best known lattices.

Construction A:

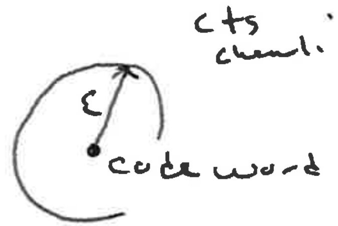
A binary code of length n is a subset ~~C of elements~~ of $\{0,1\}^n$, the binary words of length n.

elements of C are the code words.

Given a binary code C, $|C| = n$, There is a packing

$$P(C) = \{ x \in \mathbb{Z}^n : x \bmod 2 \in C \}$$

This gives some relation between packing problems in high dimensions and error detecting / correcting codes.



The centers of ε-spheres in an ε pack. correspond to code words that are distance ε apart.

dense packings → large collections of code words well separated.

~~Hamming code~~ $[\{0,1\}]^n$

~~single error detecting~~ code ~~of 3 bits,~~

Block code. (6,3)

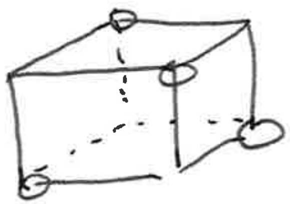
101 → 101101

repeat pattern.

parity check. (mod 2). (3,2)

11 → 110

can see this on Hamming cube.



000
01ϕ
101
110

all have hamming dist 2,

2 bits apart.

in general, then.

parity check codes ⟷ D_n

even coordinate lines.

binary codes

⟵ ⟶ Construction A

lattices.

lattice ⟺ e linear code

that is linear comb of codewords

are code words...

Hamming codes

give $\underline{E_7}$ and E_8

So in dim 10,
there is a special Gray code.

C_{10}, which is unique, and
is quite dense.

It is based on a Gray
code or reflected binary code.

~~Constructed case.~~

Gray codes are maps.

$$(\mathbb{Z}_2)^n \longrightarrow (\mathbb{Z}_2)^n$$

where the successor of
an element differs by
a single bit flip.

binary		Gray.
0 0	0	0 0
0 1	1	0 1
1 0	2	1 1
1 1	3	1 0

~~[illegible]~~.

replacement procedure to
count both binary
and Gray codes

Exercise. binary $\Leftrightarrow$ gray
invent your gray code.

In dim 10. $\left[\begin{array}{l}\text{Binary codes with} \\ \text{Minimum dist at least 4}\end{array}\right]$

we will construct C_{10}

as follows.

$\{a, b, c, d, e\} \in (\mathbb{Z}_4)^5$

Satisfying relations:

$b\ c\ d \in \{\cancel{0}, \pm 1\}$

$a = c - d$

$e = b + c$

and all cyclic ~~[illegible]~~ ~~the~~ permutation then of.

Then apply the Gray map.

$0 \to 00$

$1 \to 01$

$2 \to 11$

$3 \to 10$.

$8 \cdot 5$ code words.

$(2^3 \times |\cancel{\mathbb{Z}}_5|)$

length 10.

Claim:- Ham dist at least 4. Exam...

Check...

AC_{20} optimal for dim 10 LP bound $\Rightarrow$ < 40.

6.5)

An aside about lattices and non-lattice packings ~~~~

for other objects ...

$aab = 1$

Ellipsoid packings.

$$\begin{bmatrix} 1 & & \\ & a & \\ & & a^2 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} a \\ a \\ b \end{pmatrix} \to \begin{pmatrix} 1 \\ a \\ a^2 \end{pmatrix} \begin{pmatrix} a \\ a^2 \\ 1 \end{pmatrix}$$

polygon packings.

HCB

Bravais packing

density jump results..

ex in 2d ... 1d..

(M)

Projects and Quiz

Ideas:
- Error correcting codes - Infinite things
- other sphere packing bounds
- Entropy of jammed systems
- Translation of Research of old paper stuff ...

Format for Quiz

> definitions
> Short answers
> Sketches

Last hour of week
back ...

I need to schedule some dates still ...

Fri: 13.5.2016

HS BE01 Steyrergasse 30 EG

14:00