Lecture 7)

a short introduction
to configuration space.

A configuration space is a topological space that records the possible configurations of an object we wish to study.

For example,

The 2^N possible ~~coin~~ results of ~~flips~~ N coin flips.

The tilings of a checkerboard by dominos

The ~~~~ ~~Config~~ positions + orientations of a polytope in space.

The ~~~~ mechanical configurations of a linkage.

The positions (and velocities) of particles in a box.

This last one is more closely related to what we might want to study, relative to the first few lectures.

ie positions of points in space.

There are the classical configuration spaces ~~of~~ of N labelled distinct points in a topological space X, ~~statistical mechanics~~,

~~also study~~

$$\boxed{C^n(X) \text{ or } \mathrm{Conf}(N,X)}$$

$$\mathrm{Conf}(N,X) = \prod_{i=1}^{N} X - \Delta$$

where $\Delta = \{(x_i)_{i=1}^{N} : x_i = x_j, \ i \neq j\}$ for some

$=$

Remark / Exercise

Think about how this might be related to the Weyl Group associated to A_n

$=$

There are also unlabelled version

$$\overline{\Sigma}_{M}$$

and when X is endowed with a metric, we can consider reduced version r.

$$\overline{\mathrm{Isum}(X)}$$

So the unlabelled version chooses a single representative class for permutation of indices

and the reduced version reduce by symmetries of ambient space.

Ex. $\mathrm{Conf}(N, \mathbb{R}^1)$

$= \mathbb{R}^N \setminus \Delta$

unlabelled $\rightsquigarrow$ $\{x_1 \neq x_2 \neq x_3 \ldots\}$.
can choose to work inside space...

reduced ...
can shift some x_i to
0 ... for example
to smallest ..

$x_1 = 0$ ~~and~~ $x_1 < x_2$ $\cap$ $x_2 < x_3$ ...

Or for $\mathrm{Conf}(N, S^2)$

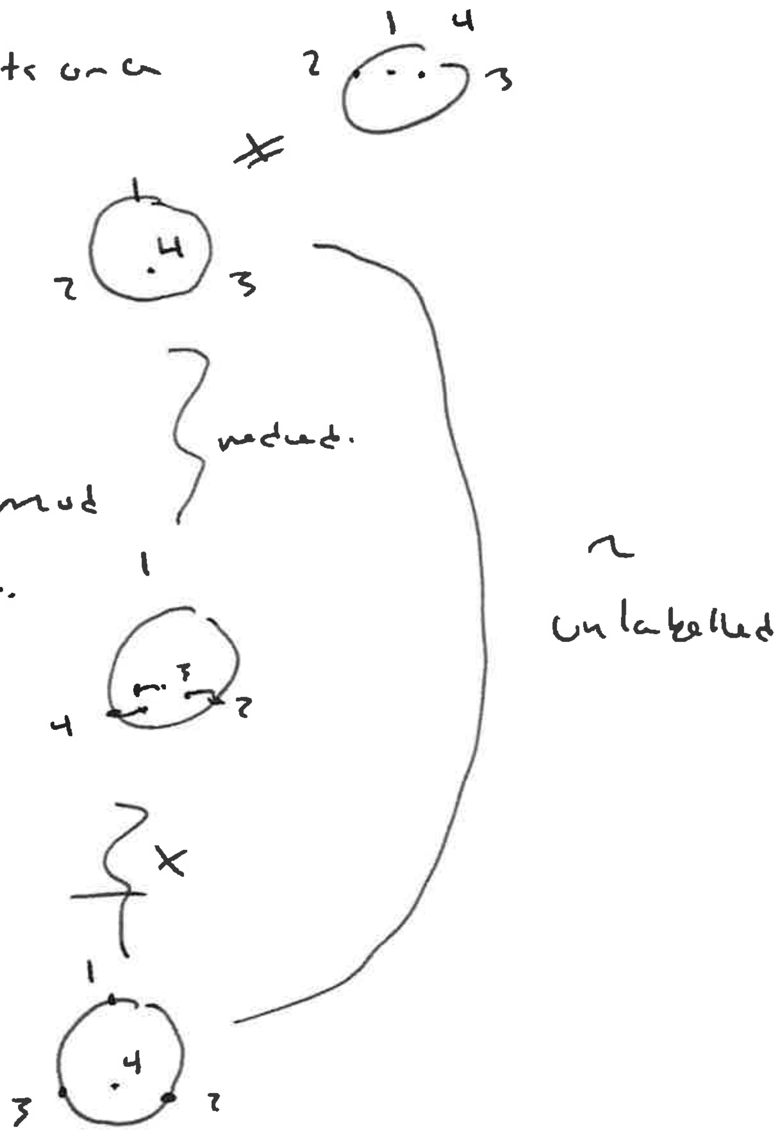
labelled distinct pts on a sphere.

reduced sphere.

~~Conf...~~

points on S^2 mod

ambient rotation.



7.2)

These spaces are ~~~~ related to the Artin Braid groups, which we may define as:

$$\pi_1(\mathrm{Conf}(N, X)/\Sigma_n)$$

(the braid group)

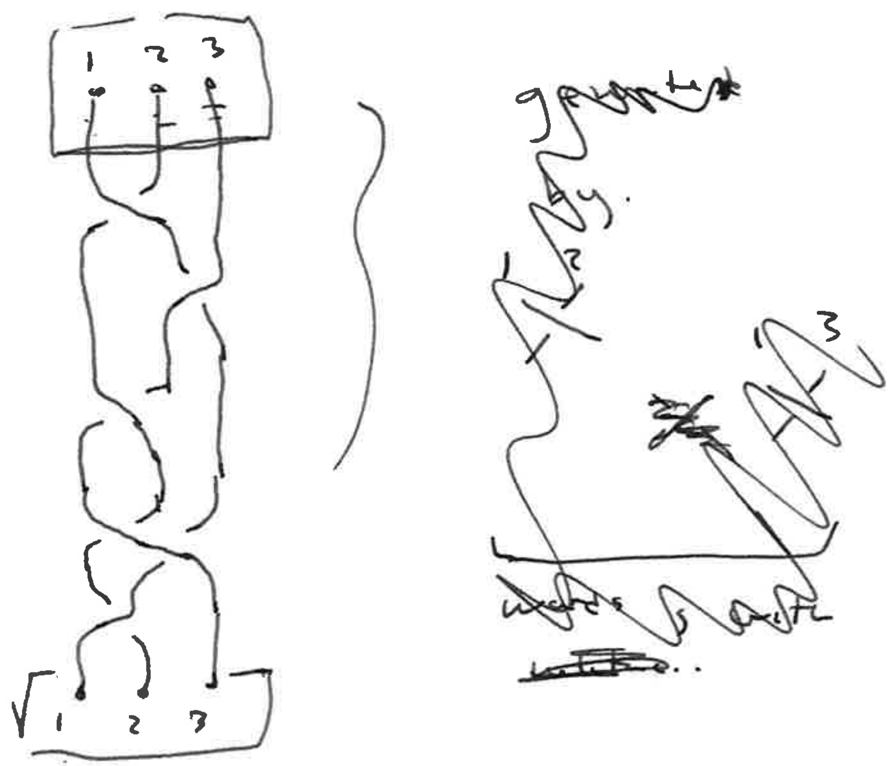
and

$$\pi_1(\mathrm{Conf}(N, X))$$

The pure braid group of X

To see why these are called braid groups, consider what a path is in $\mathrm{Conf}(N, \mathbb{R}^2)$

x_0 (base pt)



7.3) To configurations, we would like to associate various functions, based on what the model.

For examples an electrical potential, a potential sum of what... a hard shell potential..

which can describe different energy levels and also constrain possibly systems..

We may ask about

Minima, maxima, (local or global)
(critical points)
(generic configurations)

This breaks down into a couple categories..

Small N ← by hand

large N ← by statistical engines or by reasoning..

Classic examples for large N ... Expected value of N coin flips. $\{0,1\}$

⟶ central limit thm.

So for example...

N particles

K states

E_i $i \in \{1 \ldots K\}$ energy associated to that state.

Closed system so ~~E~~ constant

Satisfies:

$$E = \sum_{i=1}^{K} E_i N_i$$

$$N = \sum_{i=1}^{k} N_i$$

Zoom...

Next time.

Number of ways to place N particles in a partition $\{N_1 \ldots N_K\}$

$$W = \frac{N!}{N_1! \ldots N_K!}\text{,}$$

optimal partition dominates...

want to determine the partition that maximizes W

Can derive Boltzmann max. as.

$$\frac{N_i}{N} = \frac{e^{-\beta E_i}}{\sum_{j=1}^{k} e^{-\beta E_j}}$$

$$\Pr(\text{state } i) = \frac{e^{-\beta E_i}}{Z}$$

where $Z = \sum_{j=1}^{K} e^{-\beta E_j}$, $\beta = \frac{1}{kT}$

partition function. constant..

Note, $T \to \infty$

$$Pr(i) \to \frac{1}{K}$$

$T \to 0 \Rightarrow$

$Pr(i) \to$ minimal energy ..

But lets consider another sketchy physics analysis of the distribution of gas in a box.
(non interacting, static).
simple.

2N points

$$\mathrm{Conf}(2N, [-1,1]^3)$$

probability that the first coordinate > 0 for $N+m$ particles

P_m.

2^{2N} choices for condition.

and $\binom{2N}{N+m}$ particles with 1st coordinate > 0

(This is really the coin flip analysis).

So $$P_m = 2^{-2N}\left(\frac{2N!}{(N+m)!(N-m)!}\right)$$

We use a very bad version of Stirlings approx.

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \sim \left(\frac{n}{e}\right)^n$$

Then $$P_m \approx 2^{-2N}\left(\frac{2N}{e}\right)^{2N} \Big/ \left[\left(\frac{N+m}{e}\right)^{N+m}\left(\frac{N-m}{e}\right)^{N-m}\right]$$

$$= (N)^{2N} \Big/ \left[(N+m)^{(N+m)}(N-m)^{(N-m)}\right]$$

$$= \left(1+\frac{m}{N}\right)^{-(N+m)}\left(1-\frac{m}{N}\right)^{-(N-m)}$$

$$= \left(1-\frac{m^2}{N^2}\right)^{-N}\left(1+m/N\right)^{-m}\left(1-m/N\right)^{m}$$

$$\overset{m \ll N}{\approx} \left(e^{-m^2/N^2}\right)^{-N}\left(e^{m/N}\right)^{-m}\left(e^{-m/N}\right)^{m}$$

$$\approx P_0 e^{-m^2/N}$$

Crude

Central limit thm

~~Pr~~

$$\sigma^2 = \frac{N}{2}$$

So $\sigma \approx \sqrt{\frac{N}{2}}$.

Expt

$\Rightarrow$ for large N,

95% are within

of expected number

$2\sqrt{\frac{N}{2}}$

and

$$1 = \sum_{m=-N}^{N} \approx \int_{-\infty}^{\infty} P_0 e^{-\frac{m^2}{N}} dm = P_0 \sqrt{\pi N}$$

$$\Rightarrow \quad P_0 = \frac{1}{\sqrt{\pi N}} . \qquad P_m \approx \sqrt{\frac{1}{\pi N}} e^{-\frac{m^2}{N}}$$

~~So The sign of the first candidate~~ ~~count of the~~

~~density fn~~

So the number of points density fn

the expected number of points ~~from~~

N with positive 1st candidate

is gaussian curve

So if we pick a point in high (so we get far out a while...)

dimensions $\Rightarrow$ $[-1,1]^{3n}$

~ (half of the cube to the point, half right...)

just like the config... makes sense...

we can also look at energy... etc... quick configurations this way..

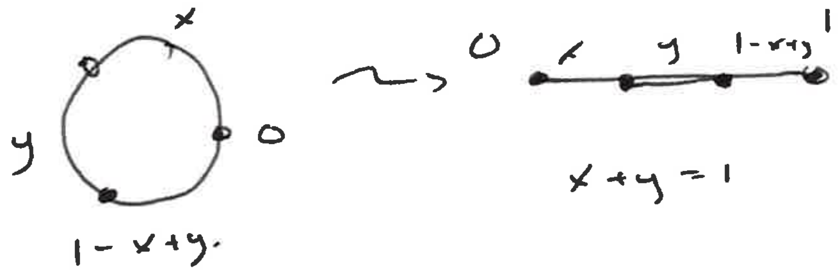
But we may be interested in smaller configurations like nearest neighbor here...

so Stirling's approximation is bad...

How do we analyze such things?

Lets look at pts on S^1

Redud $(3, S^1)$



in gen'l for N pts on
have a simplex..
with.

We can attach injectivity radius fcn..

$$\rho = \min\left(\tfrac{1}{2}\,|\mathrm{geodesic}(x_i, x_j)|\,,\ i\neq j\right)$$

[Remark:... conceptual aside unit of injectivity rad.. e.g. its a (box-circle)...]

Lecture 8)

(Smooth) Morse Theory

8.1) We started to talk about spaces $\mathrm{Conf}(N, X) : \prod^{N} X \setminus \Delta$

with

$$\Delta = \{ (x_i)_{i=1}^{N} : x_i = x_j \text{ for some } i \neq j \}.$$

Examples:

$\mathrm{Conf}(3, S^1)$

$\approx$ 

$\mathrm{Conf}(N, \mathbb{R}^1)$

$\to$ " A_{n-1} " Weyl chamber

Consider the Affin. space $(x_i)_{i=1}^{n}$

$V = \sum x_i = \text{const.}$

Then in these subspaces the hyperplanes $x_i = x_j \cap V$ are the borders of the Weyl chambers in our constructe of A_{n-1} root systems ~~of~~ ~~subspace~~ in $\mathbb{R}$ $V \subset \mathbb{R}^n$

for $n = 2$ we had.

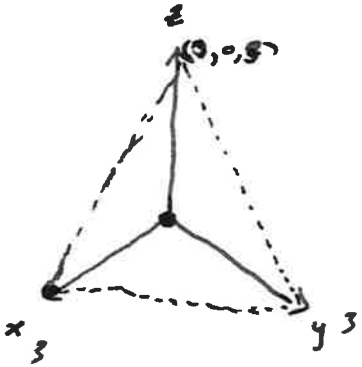
Simple roots.

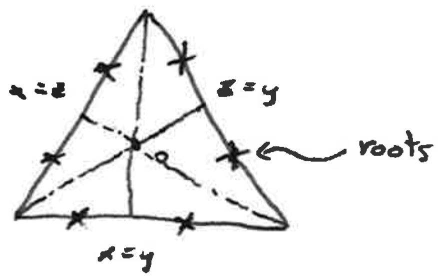
$$\left\{ \begin{matrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{matrix} \right\}$$

So the root system is given by. [const=3]

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

generated by swapping coords. ie reflecting through the $x_i = x_j$ $i \neq j$ hyperplane $\Big\}$ $+(1,1,1)$





~~These (smooth) manifolds~~ are.

We will

Consider spaces M, a Hausdorff, 2nd Countable topological space.

topology that sep elts and a countable basis for the topology.

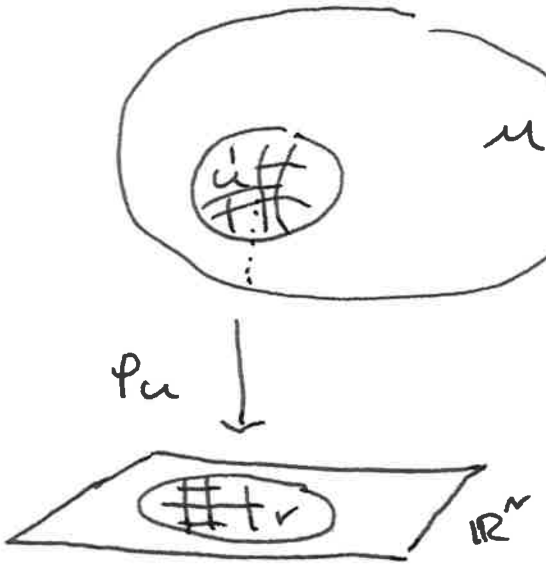
(basically to eliminate pathological examples)

Def. a homeomorphism $\varphi: M \to N$ is a continuous bijection with cts inverse.

- $\varphi: M \hookrightarrow\!\!\!\!\rightarrow N$
- $U \subset M$ open $\iff \varphi(U) \subset N$ open

A coordinate chart for a ~~[illegible]~~ space M is a homeomorphism from a coordinate neighborhood U to $\mathbb{R}^n$.

$$\varphi_U : U \subset M \longrightarrow V \subset \mathbb{R}^n$$



A smooth bijection with smooth inverse is a diffeomorphism.

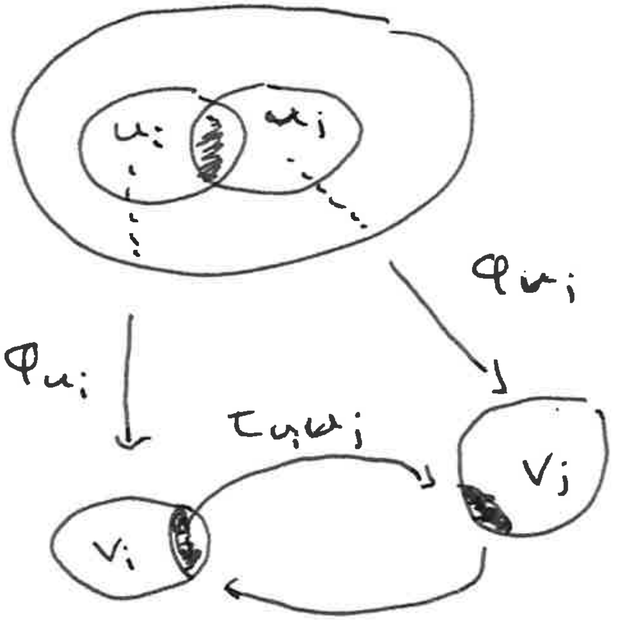
So in the smooth setting, coordinate charts should be diffeomorphisms.

~~Then an atlas / maximal smooth differentiable structure.~~

~~on a manifold~~

A manifold is a space M with a collection of charts (atlas) $\{\varphi_{U_i}\}$ that ~~[illegible]~~ indexed by ~~correspond to~~ a cover of M $\{U_i\}$ and transition maps that patch charts together.

E_{ij}



$\tau_{U_jU_i} = \varphi_{U_i}\varphi_{U_j}^{-1}$

on $\cap$

A smooth differentiable structure is a collection of coordinate charts (smooth)

$$\varphi_i : U_i \to V_i \subset \mathbb{R}^n$$

$$U_i \subset M$$

st $M = \cup U_i$

$\tau_{ij} = \varphi_j \varphi_i^{-1}$ is smooth.

and it is a maximal atlas.. ie if a chart is compatible with all φ_i as above... it is in the collection.

Basically ignore this for computation as they "unique."

So we have a topological space with a smooth structure, a smooth manifold (of dim n)

8.2) Examples.

- $\mathbb{R}^n$, choose the identity map on U_i
- $\mathbb{C}^n \to \mathbb{R}^{2n}$ with $(\mathrm{Re}, \mathrm{Im})$ on U_i
- $S^n \to \mathbb{R}^n$ stereographic projection on $S^n \setminus \{N\}$ and $S^n \setminus \{S\}$
- on tori by products of S^1 maps

...

- on open subsets of manifolds by considering the subspace topology.

We characterize smooth maps between manifolds.

$f : M \to N$: f for each $p \in M$ and some charts $(\Rightarrow \forall)$

φ, ψ for M, N

$p \in U \subset M$ $\quad$ $f(p) \in V \subset N$

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"'] & V \arrow[d, "\psi"] \\ \varphi(U) \arrow[r, dotted] & \psi(V) \end{tikzcd}$$

The induced map $\psi f \varphi^{-1}$ is smooth on its domain.

Similarly for maps ~~to~~ to other manifolds etc

So we can define (diffeomorphisms) of homeomorphisms manifolds.

That is a (smooth) bijection

$M \to N$

with (smooth) inverse

$M \cong N$

8.3) ~~Morse Theory~~

~~A way to study (smooth) manifolds and (smooth) functions on them.~~

8.3 ~~also~~ We will want to talk about homotopy type as well, since we may be dealing with things that are not diffeomorphic, but topologically "similar".

We say two maps

$f, g : M \to N$ are homotopic

if there exists a ~~homotopy~~ a cts function.

$h : M \times I \to N$

s.t. $H(x,0) = f(x)$

$H(x,1) = g(x)$

$f \simeq g$

M and N are of the same homotopy type

if there are maps:

$f : M \to N$

and $\qquad$ st $\quad f \circ g \simeq \mathrm{id}_N$

$g : M \to N \qquad\qquad g \circ f \simeq \mathrm{id}_M$

$$f \simeq g \qquad \begin{tikzcd} M \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\simeq"] & N \end{tikzcd}$$

if maps f, g.

$f : M \longrightarrow N$

$g : M \longrightarrow N$

are homotopic if

There is a cts function.

$$H : M \times I \longrightarrow N$$

st
$$H(x,0) = f(x)$$
$$H(x,1) = g(x)$$

two spaces. M and N are (smooth) homotopic (same homotopy type) if there are maps

$$f: M \to N \qquad \text{st} \qquad f\circ g \simeq id_N$$
$$g: M \to N \qquad\qquad g\circ f \simeq id_M$$

Idea, ~~one space can~~ spaces can be ctsly (smoothly) deformed to each other.

$$\begin{tikzcd} \mathbb{D}^2 \arrow[r, bend left, "f \text{ radial retract} \to 0"] & \{0\} \arrow[l, bend left, "\text{inclusion } g"] \end{tikzcd}$$

$$h(t,(r,\theta)) := ((1-t)r, \theta))$$

$$g\circ f \begin{cases} \mathbb{D}^2 \to \mathbb{D}^2 & id \\ \quad \simeq \; h: \\ D^2 \xrightarrow{f} 0 \end{cases}$$

$$f\circ g = id\,\{0\}$$

8

Homotopy relative to a subspace:

is a homotopy which fixes elts of of a subsp.

$f, g : M \longrightarrow N$

$K \subset M$

f, g homotopic relative to K

if there is ~~hkt~~ homotopy h betw. f and g.

$h : M \times I \longrightarrow N$

s.t.

$h(k, t) = f(k) = g(k)$

$\forall \; k \in K, \; t \in I.$

If $f = id$

g retraction.

cts $\begin{cases} \cancel{M} \longrightarrow K \\ g|_K = id. \end{cases}$

this homotopy defines a strong deformation retract of M to K.

Ex. last $D^2 \to \{0\}$ example.

§. 8.4) An aside about homotopy groups

homotopy classes of maps. (to pointed spaces)

Consider $f : S^1 \longrightarrow (X, x_0)$

$\rightsquigarrow$ paths. $I\, \cancel{[0,1]} \longrightarrow X$

st $f(0) = f(1) = x_0$

then these have a group structure under concatenation

$$f \cdot g = \begin{cases} f(2t) & t \in [0, \tfrac{1}{2}] \\ g(2t-1) & t \in [\tfrac{1}{2}, 1] \end{cases}$$

This is $\pi_1(X, x_0)$

~~π_1 = homotopy~~

In general one can consider maps.

$$f : S^n \to (X, x_0)$$

as. $\cancel{[0,1]^n}$. maps. st. $I^n \to X$

$f(\partial[I^n]) = x_0$.

with concatenation of ~~a compt~~.. defined appropriately.

If X is path connected we may ignore the base pt x_0.

Draw $(\mathbb{D}^2_{1\times 0}) \{ \cancel{\mathbb{x}}\, 0 \}$

Example.

~~just~~ note for

~~Note~~

We will not deal with this too much ~~use~~), but note this is a nice [illegible] for spaces that are homotopy equiv.

Then

$U \ni p \in M \qquad V \ni f(p) \in N$

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"'] & V \arrow[d, "\psi"] \\ \varphi_\alpha U \arrow[r] & \psi_\alpha V \end{tikzcd}$$

$\psi f \varphi^{-1}$ is smooth

on domain.

---

then we can define a

diffeomorphism of manifolds.

~~Sadly, not all the maps we want to work with will be smooth,~~

If we change the underlying conditions ... the maps ...

we can still define coordinate charts, atlases ..

etc ...

8.2 Morse theory.

Morse theory is a fairly gentle way to study manifolds and functions on them.

The idea is ~~basically~~ that the structure of the sets of critical points of functions and the topological structure of the manifold are related. (~~see~~ as follows Milnor + others)

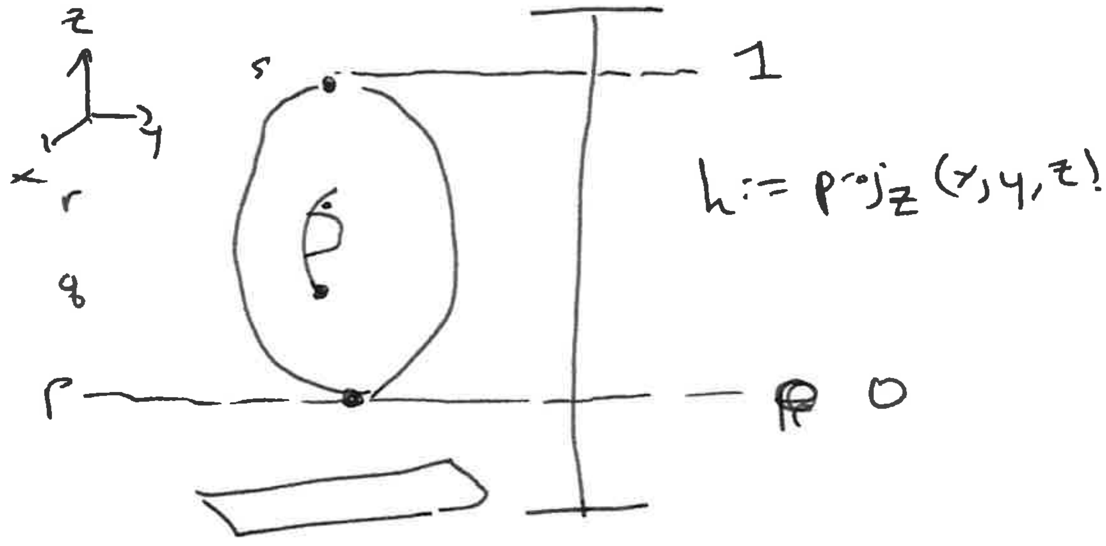
Classic picture... consider f

$$T^2 \xrightarrow{f} \mathbb{R}$$

given by the following.

$$T^2 \hookrightarrow \mathbb{R}^3 \longrightarrow \mathbb{R}$$

but try also following Milnor ...



Consider the sublevelsets:

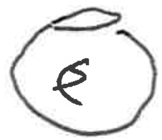
$$M_a := \{ x \in M : f(x) < a \}$$

$a < 0 = f(p)$ $\quad$ $M_a = \emptyset$

$f(p) < a < f(q)$ $\quad$ M_a is a 2 cell i.e a 2 ball

$f(q) < a < f(r)$ $\quad$ M_a is a cylinder.

$f(r) < a < f(s)$ $\quad$ M_a is a torus $\setminus$ disc



$f(s) < a$ $\quad$ M_a is a torus.

There are handle or cell attachments

attaching a k-cell

to a topological space ~~X~~ Y

e^k is a closed k dim ball

with $\partial(e^k) = S^{k-1}$

g cts. $S^{k-1} \to Y$

$Y \cup_g e^k$ is the

quotient of attaching a

k cell to Y by g

defined by.

$Y \sqcup e^k$

and identifying. $\sim$

$x \in S^{k-1}$ with $g(x)$

$[x \in S^{k-1} \sim g(x)]$

K

The homotopy type

(and the diffeomorphism type

on a nbhd of a neighborhood)

change exactly at the

critical points of the function

f... these points where

~~the Hessian is~~ gradient is 0.

$$\nabla f = 0$$

If we look at

the Hessian, (non-degenerate)

we see.

Hf_p has signature $(2,0)$

$\operatorname{sig}(Hf_q = Hf_{(r)}) = (1,1)$

$\operatorname{sig}(Hf_s) = (0,2)$.

we call the ~~negative~~ negative part

the index, ie the #

decreasing directions from the

critical point and see

it corresponds to the dimension

of the attached cell.

This is why we want to start with smooth functions...

we will call a smooth function **Morse** if it has no nondegenerate critical pts

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we will formalize this next time...