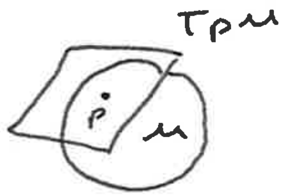
Lecture 9: Morse functions and their critical points.

Last time, we defined a function (smooth) $f: M \to \mathbb{R}$ to be <u>Morse</u> if it had no non-degenerate critical points.

9.1. What does this mean?

We need to understand criticality so we introduce the Tangent space T_pM at a point $p \in M$.



We will consider velocities of curves

$$\gamma: [-1,1] \to M$$
$$\gamma(0) = p$$

and with respect to a chart $\varphi: \cancel{} \to \mathbb{R}^n$

$U \subset M$
$\cup$
p

$$(\varphi \circ \gamma): [-1,1] \to \mathbb{R}^n$$

differentiable at 0

The classes of curves identified by their tangents ~~vector~~ ~~vector~~ at 0 define the tangent space at p

Given a smooth map.

$f: M \to N$, there is

an induced linear map.

$$f_*: T_pM \to T_pN$$

$$f_*(\gamma'(0)) = (f \circ \gamma)'(0).$$

(The differential or pushforward.)

$f: M \to \mathbb{R}$ smooth, p is a

critical point of f if

$$f_*\ T_pM \to T_{f(p)}\mathbb{R} = 0$$

$f(p)$ is a critical value

In local coordinates for T_pM

This is written.

$$\frac{\partial f}{\partial x_i}\Big|_p = 0 \quad \forall\, i = 1 \dots n$$

or

Also, there is a bilinear form.

which can be written similarly.

$$\frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_p = H_f(p)$$

or $f_{xx}(p)$

The Hessian of f at p.

This is non degenerate if

the matrix is non-singular.

The notion of index is well defined in this case.) # of negative eigenvalues.

6.2) Why do we care about non-degenerate critical points?

Morse Lemma.

If p is a non-degenerate critical point for f smooth, there is a local coordinate system about p st

$$f = \sum_{i=1}^{\lambda} -x_i^2 + \sum_{j=\lambda+1}^{n} x_j^2$$

where λ is the index of f at p.

Proof (Sketch)

We may shift s.t.

$$f(p) = 0$$

FTC and Integral Remainder Taylor Thm. gives

$$f(x_1 \dots x_n) = \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx_1 \dots tx_n)\, x_i\, dt$$

...

and in fact

$$f(x_1 \dots x_n) = \sum_{i,j} h_{ij}(x_1 \dots x_n) x_i x_j$$

for some functions h_{ij} smooth...

and in fact . Can symmetric and can.

$$h_{ij} = \frac{1}{2} \frac{\partial f}{\partial x_i \partial x_j} (0)$$

in some coordinate system.
for T_pM.

This can be inductively diagonalised in a nbhd of p, for example via the spectral theorem + continuity argument.

Ex in dim 2.

$$f(x,y) = x^2 h_{11} + 2xy h_{12} + y^2 h_{22}$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{0,0} = 2h_{11}(0,0)$$

etc...

by a coordinate change

$h_{11} \, (0,0) \neq 0$

cts $\Rightarrow$ $\exists$ nbhd st $h_{11} \neq 0$.

Then we can define a local change of coordinates.

$$\bar{x} = \sqrt{|h_{11}|} \left(x + \frac{h_{12}}{h_{11}} y \right)$$

$$\Rightarrow \bar{x}^2 = |h_{11}| \left(x^2 + 2 \frac{h_{12}}{h_{11}} xy + \frac{h_{12}^2}{h_{11}^2} y^2 \right)$$

Then for case $h_{11} > 0$

$$\bar{x}^2 = h_{11} x^2 + 2 h_{12} xy + \frac{h_{12}^2}{h_{11}} y^2$$

Since

$$f = \underbrace{x^2 h_{11} + 2xy\, h_{12}} + y^2 h_{22}$$

$$\bar{x}^2 - \frac{h_{12}^2}{h_{11}} y^2 + h_{22} y^2$$

proofs to sum for other cases, higher dimensions.

Cor: non degenerate critical points are isolated.

6.3 A One parameter subgroup of diffeomorphisms:

of a manifold M is a smooth map

$\Phi \quad \varphi: \mathbb{R} \times M \longrightarrow M$

such that: for each $t \in \mathbb{R}$

$\Phi \quad \varphi_t: M \to M$ defined as $\underset{\Phi}{\varphi_t}(p)$

$= \underset{\Phi}{\varphi_t}(t, p)$ is a diffeomorphism

of M onto M <u>and</u>

for all $t, s \in \mathbb{R}$ $\quad \underset{\Phi}{\varphi_{t+s}} = \underset{\Phi}{\varphi_t} \circ \underset{\Phi}{\varphi_s}$

This defines a vector field

X on M [for smooth (checkmarked) $f: M \to \mathbb{R}$ by

$$X_q(f) = \lim_{h \to 0} \frac{f(\varphi_h(q)) - f(q)}{h}$$

So this is an assignment of a function to each point q of M ... so it takes a directional derivative of f ...

. X <u>generates</u> the map φ.

Lemma

A smooth vectorfield on M which vanishes outside of K compact $\subset M$ generates a unique 1p grp of diff. of M.

Proof: consider smooth curve

$\gamma : t \mapsto \gamma(t) \in M$

with velocity $\frac{\partial \gamma}{dt} \in T_{\gamma(t)} M$

~~defines~~ Characterised by:

$$\frac{d\gamma}{dt} f = \lim_{h \to 0} \frac{f(\gamma(t+h)) - f(\gamma(t))}{h}$$

Let $\overset{\Phi}{\varphi}$ be a 1p grp of d. generated by a vectfield X as in the Lemma.

Then for fixed q

$$t \mapsto \overset{\Phi}{\varphi}_t(q) \text{ satisfies:}$$

$$\frac{d \overset{\Phi}{\varphi}_t(q)}{dt} = X_{\cancel{\varphi_t(q)}}$$

$$X_{\cancel{\varphi_t(q)}}$$

$$X_{\varphi_t(q)}$$

$$\overset{\Phi}{\varphi}_0(q) = q$$

Since.

$$\frac{d\varphi_t^{\Phi}(q)}{dt}(f) = \lim_{h\to 0} \frac{f_{t+h}(q) - f(\varphi_t(q))}{h}$$

$$= \lim_{h\to 0} \frac{f\varphi_h^{\Phi}(\varphi_t q) - f(\varphi_t^{\Phi}(q))}{h} = X^{\Phi}_{\varphi_t(q)}(f)$$

Exist uniq of soln.. (Picard-Lindelöf applied generally..).

has a unique solution

$\Rightarrow$ that smoothly depends on initial condition..

$\Rightarrow$ for all $q \in M$, there is a neighborhood U

and $\varepsilon > 0$ st $\frac{d\varphi_t^{\Phi}(q)}{dt} = X^{\Phi}_{\varphi_t}(q)$ $\varphi_0^{\Phi}(q) = q$

has a unique smooth solution for $q \in U$

$|t| < \varepsilon$

Since X is compact

Since K is compact, there is a minimal $\varepsilon > 0$

for such neighborhoods with $\ldots$ K.

~~and for~~

for $q \notin K$, let $\Phi_t(q) = q$

$\Rightarrow$ There is a unique solution $\Phi_t(q)$ $t < \varepsilon$ smooth (in t and q)

for all q in M.

note. $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for $|t|, |s|, |t+s| < \varepsilon$

So Φ_t is a diffeo for $|t| < \varepsilon$

but we can iterate this process by LNP...

for $t > \varepsilon$ by. $\lfloor \frac{t}{\varepsilon} \rfloor \varepsilon + (t - \lfloor \frac{t}{\varepsilon} \rfloor \varepsilon)$

steps.. $\lfloor \frac{t}{\varepsilon} \rfloor + 1$

$\boxed{\varepsilon}$

6.4 ~~Homotopy~~ Theorem: Diffeomorphism

For $f: M \to \mathbb{R}$, we have

Sublevelsets

$$M_a := f^{-1}(-\infty, a]$$

$$= \{p \in M : f(p) \leq a\}.$$

~~Given~~ If ~~Ex~~ f, smooth function

$M \to \mathbb{R}$, ~~Ex~~.

$a < b$ and $f^{-1}[a,b]$ is

compact and contains no

critical points of f

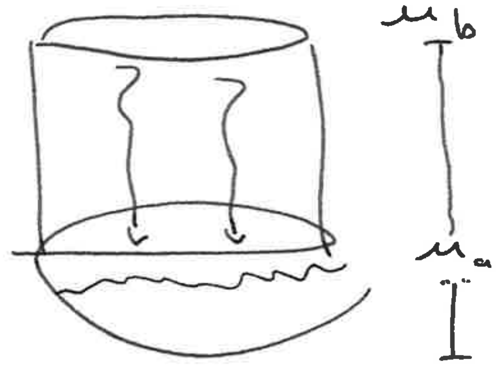
Then $M_a \cong M_b$

Idea:

$f^{-1}(b)$

$-\nabla f \downarrow$

$f^{-1}(a)$



M admits a metric...

need to define $\langle x, y \rangle$ the inner product of vectors tangent

so we can define

∇f, which is

characterized by:

$$\langle X, \nabla f \rangle = Xf$$

The directional derivative of f along X. for any vf X.

Clearly $\nabla f =$

$\nabla f \cdot v$

Note ∇f vanishes at the critical points of f.

If $\gamma : \mathbb{R} \to \mathbb{R}$ is curve with velocity vector $\frac{d\gamma}{dt}$ then.

$$\left\langle \frac{d\gamma}{dt}, \nabla f \right\rangle = \frac{d f \circ \gamma}{dt}.$$

Define a smooth function.

$$\rho : M \to \mathbb{R}.$$

that is $\dfrac{1}{\langle \nabla f, \nabla f \rangle}$ on $f^{-1}[a,b]$ 0 ~~[illegible]~~. outside of cmpt nbhd of that...

Then vf X defined as.

$$X_q = \rho(q)(\nabla f)_q$$

Satisfies the Lemma for 1p ~~set~~ groups.

(vanishes outside compact set)
smooth...

$\Rightarrow \exists$ 1p $\overset{\Phi}{\varphi_t} : M \to M$

grp of diffeo..

For fixed q, consider

$$t \to f(\varphi_t(q))$$

If $\varphi_t(q) \in f^{-1}[a,b]$

Then.

$$\frac{d f(\varphi_t(q))}{dt} = \left\langle \frac{d \varphi_t(q)}{dt}, \nabla f \right\rangle = \langle X, \operatorname{grad} f \rangle$$

$$= 1$$

$$\Rightarrow t \mapsto f(\varphi_t(q))$$

is linear with dh 1 for $f(\varphi_t(q))$ in $[a,b]$

$\varphi_{b-a} : M \to M$ is a diffeo.

$M^a \to M^b$.

furthermore..

$r_t : M_b \to M_a$

$$r_t(q) \begin{cases} q & \text{if } f(q) \le a. \\ \underbrace{\varphi_{t(a-f(q))}}(q) & a \le f(q) \le b \end{cases}$$

Then ...

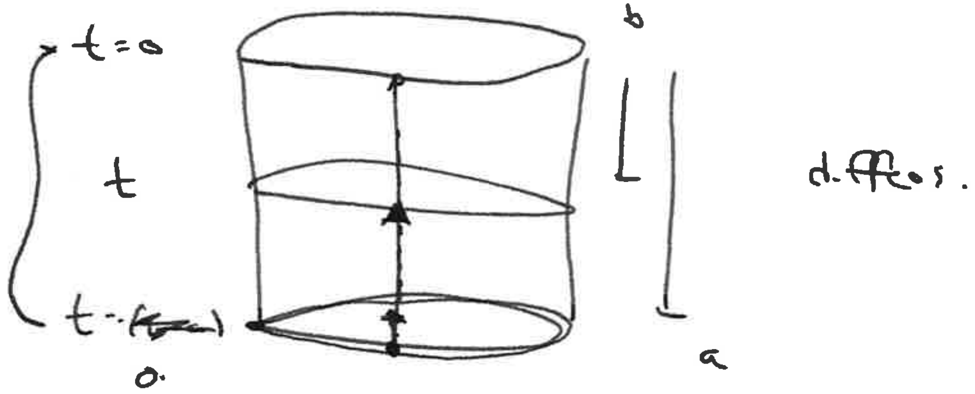
$$\begin{cases} r_0 = \mathrm{id}. \\ r_1 \text{ is a retraction of } M^b \to M^a. \end{cases}$$

retraction.

$\boxtimes$

$\varphi_{(b-a)} : M \to M$ is diffeo.

$M^a \xrightarrow{\approx} M^b$



Lecture 10: Configuration Spaces and Morse-like results.

10.1) For smooth functions we have the diffeomorphism result that says:

$f: M \to \mathbb{R}$

- $a < b$
- $f^{-1}([a,b])$ cpt
- no critical values in $[a,b]$

$\Rightarrow M_a$ diffeo M_b

need to be rewritten out of order / missing pieces

There is also a handle attachment thm (not proved) for smooth

$f: M \to \mathbb{R}$

- p - non degenerate critical point
- index γ, $f(p) = c$
- $\exists\ \varepsilon > 0$ st $f^{-1}([c-\varepsilon, c+\varepsilon])$ cpt, contains no critical pts except c

~~Then~~

$\Rightarrow M^{c+\varepsilon}$ has homotopy type $M^{c-\varepsilon} \bigcup_g \gamma$-cell

Morse type inequalities

c^r = # critical pts of index r

Then the homotopy type changes by attaching various cells of index r as we "climb" M via the morse function.

$$\Rightarrow \sum (-1)^r c^r = \chi(M)$$

and in generally can show

$$c^r \geq b_r(M)$$

the rth betti numbers of M

[or the ranks of the rth homology grp.]

The key point for us is that the ~~path~~ betti numbers can be computed for many spaces we like.

10.2. Betti numbers for B Conf(N)

- Configuration space of pts on S^2

$$B\,\mathrm{Conf}(N) \longrightarrow \mathrm{Conf}(N-1, \mathbb{R}^2)/SO(2)$$

Via Stereographic projection from

$N \geq 3$, ... we have a correspondence generally fact.

That allows us to ~~compute~~ ~~betti numbers~~ from ~~the known~~ betti numbers.

as $$P(\mathrm{Conf}\, N-1, \mathbb{R}^2) / P(SO(2))$$

where

$$P(\mathrm{Conf}(n-1, \mathbb{R}^2)) = (1+t)(1+2t)(\ldots)(1+(n-2)t)$$

$$P(SO2) = (1+t)$$

---

pictures of punctured planes and the induction

up...

---

can use this idea to describe extensions

and induction arguments for configuration spaces in general..

for example the $K(\pi,1)$ type of config space

comes from the fact ~~that the space configur~~

~~from the other~~. that such spaces

really change nicely..

So . . .

	r = 0	r = 1	r = 2 . . .	r = 3
n = 3	1	0	0	0
n = 4	1	2	0	0
n = 5	1	5	6	0
. . .	1			
	⋮			$\ddots$ (n-2)!

[illegible]

~~N=2 fact~~ 1

10.2)

A min type function
is a parametrized
family functions.

$$f(\cancel{F}\, x, p) : \cancel{F}\, X \times M \longrightarrow \mathbb{R}$$

~~F~~ X, ~~?~~ parameter space,
compact ....

For our purposes,
finite discrete.

f smooth/smoothable
for each parameter.

Our function, the injectivity radius

$$= \cancel{\min_{i,j} d(p_i, p_j) \quad i \neq j}$$

$$\cancel{\mathrm{Conf}(N) \longrightarrow}$$

$$\rho(U) = \min_{\substack{i \neq j \\ u_i, u_j \in U}} d(u_i, u_j)$$

$$\mathrm{Conf}(N) \longrightarrow \mathbb{R}$$

is of this type.

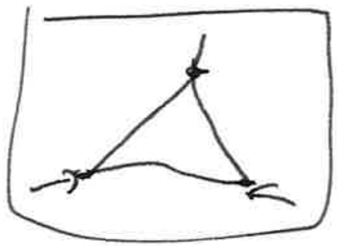
In genl, topological regularity is characterized by the non-emptiness of a ~~cone~~ intersection gradient cone. of this type.

Cor: Topology only changes in critical pts x ~~on~~ ~~cont~~ only occur when the gradient cone is empty, i.e.

$$\sum_x \omega_x df_x \Big|_{x_c} = 0 \qquad \omega_i > 0.$$

From our picture for injectivity radius, this is when gluing loses collect & is not possible.

This is

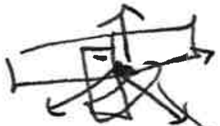


ie there is some other cone that interferes..

ie no elt in the positive span

For the case of injectivity radius

or ~~target~~ the convex cone may be empty



definitions..

$$\sum_i w_i \, df_{p_i}\big|_x = 0$$

positive vertices for all

objects...

$\mathcal{U} \in \mathrm{Conf}(N)$

with $\rho(\mathcal{U}) \geq \Theta$

Then is a contact graph.

vertices = $u_i \in \mathcal{U}$

edges $u_i u_j \Leftrightarrow d(u_i, u_j) = \Theta$

edges may be assigned a

positive weight w_i, and

if the associated graph

is force balanced --

$\Leftrightarrow$ cone is empty...

$\Rightarrow$ the only critical points for non-topologically cycles which are exactly the free boundary configurations of points..

Examples...

Thm: Suppose $N \geq 3$.

Then for ~~$d(u_i, u_j)$~~ $0 < \theta < \frac{\pi}{N}$,

$Conf(N, \theta)$ is a s def retract of ... to $Conf(N, 0)$

proof: first no balanced configurations...

Conclude by pushing....

For $N=4$...

$Conf(N,\theta)$ small

$\Rightarrow$ pitch constrained

Example $N=4$

Strategy for terms Maxima

Table .

Algorithm for searching ... by evaluation of

force configurations ...

Local Maxima ..

rings ...

rigidity ?, ~~cart~~ contact conditions

and weights ..

candidate graph sorting ..

What is this!

this is a fine handwriting.

I e for injectivity radius, it

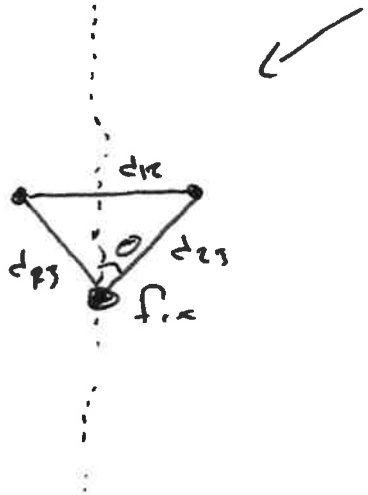
is ~~a variation in~~ the intersection

of the convex cores of motion

that increase extends between

each pair of points.

picture!: 3 points.



6 dim.

4 dim space. fix 3

3 dim space.. fix θ

given by determ..

Cone for

d_{12} is upper 12 ...

etc..

So

topological regular value <=> ~~diffeomorphic~~ ~~neighborhoods~~ ~~for the~~ neighbourhoods of value have ~~diffeomorphic~~ sub-level sets that deformation retract to each other.

So critical ~~points~~ values are where the topology changes...

For the topological regular values, this deformation retraction is defined by the ~~evolution~~ flow of some vector field defined by f, the gradient flow as in the last lecture...

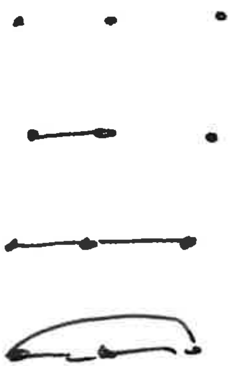
So the topology can only change at an obstruction to gradient flow.

For a min type function, this is a ~~cone~~ given by a convex cone, that is the ~~union~~ intersection of all variational cones that increase f at the point $x \in X$.

1) ~~Categori~~

Classify all ~~geometrically~~ distinct, realizable, ~~configurat~~ balanced configurations of n points on a sphere.

First graphs. 3 points.



for x. to be balanced, need value $= 2$.

 ?

 ?

~~[illegible]~~ optimal

$\boxed{\text{g-triangles}}$

geometry..

optimal