Lecture 11)

11.1) Critical Points and Jamming.

Last time we stated Thm.

ρ - regular value => elt in tangt cone

no elt in tangent cone <=> $\exists$ weights
(non-negative, not all 0) ~~(positive)~~ st the <u>contact</u>
<u>graph</u> is force balanced.

The contact graph is the ~~geometric~~
~~structure of~~ set of ~~structs~~

~~the~~ edges on S^2 between edges
exactly distance Θ apart for
$\Theta = 2\rho$

<u>Thm</u> $\mathrm{Conf}(N, 0)$

$\simeq \mathrm{Conf}(N; \frac{2\pi}{n})$

=

This also gives an
algorithm for exploring
critical / maximal configurations

ρ-evolution
update...

configuring...

11.2) Examples with picture.

$N=3$ } full picture
$N=4$

$N = \ldots$ max terms.

$N = 10$ – local max?

$N = 12$ – local max?

higher critical points.

Jitterbug?

11.3) Local maxes and standard entropy.

Ansatz of equal value
~~Eq~~: all jammed states have equal probability.

Related to the evolution of ρ flow. Also the model...

not always true...?

~~ex. cancelled~~

Boltzmann Measure

Discrete Setting:

N particles, each in one of K states.

Each state has an energy

$\mathcal{E}_i$, $i \in \{1, \dots, K\}$.

System closed so:

$$E = \sum_{i=1}^{K} E_i n_i \qquad n_i = \text{\# particles in state } i$$

$$N = \sum_{i=1}^{K} n_i$$

W, the # of ways to place particles $\{n_1, \dots n_K\}$

is the multinomial coeff.

$$\frac{N!}{N_1!\, N_2! \dots N_K!}$$

LLN $\Rightarrow$

the optimal partition dominates

for N large

determine partition

that maximizes W.

replace with $\log W$ [log monotone ↗]

$$\log W = (N \ln N - N) - \sum_{i=1}^{K} (N_i \ln N_i - N_i) + \text{err.}$$

L multip
$$\left[f = \ln(W) + \alpha\Big(N - \sum_{i=1}^{K} N_i\Big) + \beta\Big(E - \sum_{i=1}^{K} E_i N_i\Big) \right.$$

cstr. . .

$$0 = \frac{\partial f}{\partial N_i} = - \ln N_i - (\alpha + \beta E_i)$$

$$\frac{\partial}{\partial N_i}\left(N_i \ln N_i + \bar{\alpha} N_i + \bar{\beta} E_i N_i \right)$$

$$-1 - \ln N - (\alpha + \beta \varepsilon_i]$$
$$-1$$

$$-\ln N_i - (\alpha + \beta \varepsilon_i) = 0$$

$$N_i = e^{-\alpha - \beta E_i}$$

$$\frac{N_i}{N} = \frac{e^{-\beta E_i}}{\sum_{j=1}^{k} e^{-\beta E_j}} \quad \propto$$

$$Z = \sum_{j=1}^{k} e^{-\beta E_j}$$

$$\boxed{\beta = \frac{1}{T} \text{ or } \frac{1}{kT}}$$

$$\boxed{F = -T \ln Z \quad - \text{Free energy}}$$

$$\Pr(\text{state } i) = \frac{e^{-\beta E_i}}{Z}$$

$$\bar{E} = \frac{E}{N} = \sum_{i=1}^{k} \frac{E_i N_i}{N}$$

$$= \sum_{i=1}^{k} \frac{E_i\, e^{\beta E_i}}{Z} = \frac{-\frac{\partial}{\partial \beta} Z}{Z} = \underline{\underline{-\frac{\partial}{\partial \beta} Z}}$$

Lemma: $-\ln Z$ convex:

Show. $\frac{-\partial}{\partial \beta} \ln Z$ monotone.

ie $\frac{\partial^2}{\partial \beta^2} \ln Z > 0$ Why?

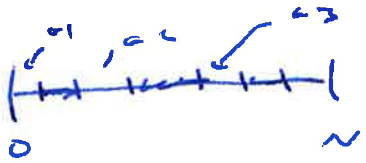
$\rightarrow$ write as variance of $E > 0$.

$\frac{1}{\beta} \ln Z := $ Free energy.

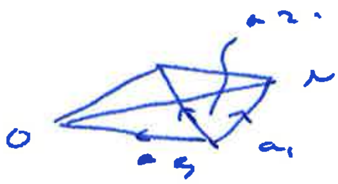
$T \rightarrow \infty \quad Pr(i) \rightarrow \frac{1}{k}$

Hard disc model.

interval.



$$\Omega_{N,3} = \{ (a_1, a_2, a_3) \quad 0 \le a_1, a_2, a_3, \; a_1 + a_2 + a_3 \le N-3.$$



$$\text{Volume } \Omega_{(N,3)} = \frac{(N-3)^3}{3!}$$

in general.

$$\text{vol } (\Omega_{N,k}) = \frac{(N-k)^k}{k!}$$

Give energy E_0 per particle.

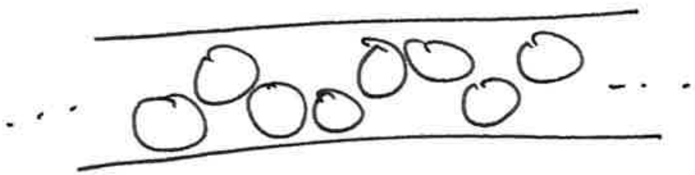
$$z = e^{-\beta E_0} \quad \text{fugacity.} \longrightarrow \underline{\underline{\text{define}}}$$

$$Z_N(z) = \sum_{K=0}^{N} z^K \, \mathrm{vol}(\Omega_{N,K})$$

$$= \sum_{K=0}^{V} \left(\frac{(N-K)^K}{K!} z^K \right.$$

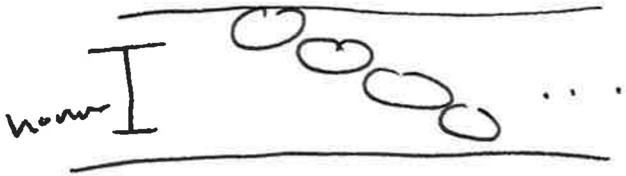
11.4) Quasi-1D model with open ends.

~~Model~~. narrow channel : combinatorial analysis.



← Jammed.
local maximal density.

Certainly saddle



length is t.

disks $r=1$ channel $2 < \mathrm{diam}_{\text{channel}} < 2+\sqrt{3}$.

Combinatorics: how many maxima for t?

The length is determined by # of adjacent

and opposite balls.

(1) (0)

Counting is # of binary words with no

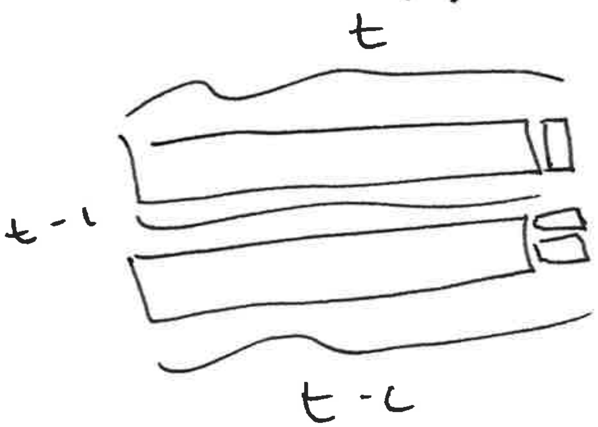
adjacent 1's

This is eq to a tiling domino problem. or stars and bars.

sequences formed of... 0 01

This satisfies the fibonacci recursion. (modulo up/down symmetry)

~~words ending in 0 of length ℓ~~



and

1, 2 ...

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given exactly k 1s in a word of length ℓ,

$\Rightarrow$ inserting k 1s into a sequence of $\ell - k$ 0's. , 0 , 0 , 0 , ... 0 ,

So into $\ell - k + 1$ slots.

So for each k => $\binom{l-k+1}{k}$ choices of order

So if length of a "1" is 2

and a "0" is L

we have

the $t = l = k \cdot 2 + (l-k)L \underbrace{+2}_{\text{step.}}$

we can also compute the max* of such quantities.

$$\frac{\binom{n-k+1}{k}}{[illegible]} \longrightarrow \frac{k}{n} \longrightarrow \frac{5}{10} - \frac{\sqrt{5}}{10}$$

And there is a "unique" dominant pretty clear.

... on last line as well...

Ex. law of large approx

e.g.

$\longrightarrow$ typical configuration looks like

$$\frac{1}{2} \pm \frac{\sqrt{5}}{10} \quad (15) \quad (\text{e.g.})$$

(compute coin flips ...) $\longrightarrow$ $\left(\frac{1}{2}\right)$

pairing values of critical pts

flow to joint asymptote ...

precisely marginal since critical

scaling should be near 0 pairs

$\Longrightarrow$

? compute the values!

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also.. for large N

so studying low order exp.